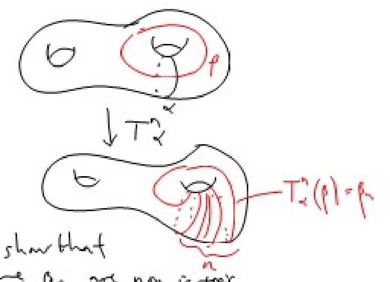
Exercise: $T_\alpha \in \mathrm{Homeo}_0(S)$ iff

α is either inessential or peripheral.

[Hint: use intersection #].

Picture:



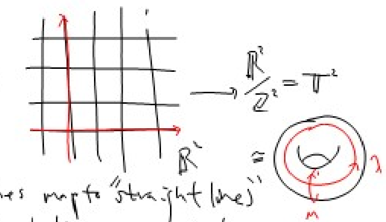
Can now show that all of the ρ_n are non-isotopic

Today: Classify elements of $\mathcal{S}(\mathbb{T}^2)$

[ess. loops on the torus]

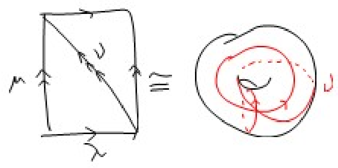
Recall:

$\mathbb{Z}^2 \hookrightarrow$



Rmk. Straight lines map to "straight lines" and rationally sloped lines map to loops.

We add a diagonal



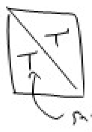
Note that (λ, μ, ν) have slopes $\left(\frac{0}{1}, \frac{1}{0}, \frac{-1}{1}\right)$

Suppose that $\rho \in \mathcal{S}(\mathbb{T}^2)$ is any given loop.

Let $\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} i(\lambda,\rho) \\ i(\mu,\rho) \\ i(\nu,\rho) \end{pmatrix} \in \mathbb{Z}^3_{\geq 0}$.

Rmk: If $(p,q,r)=\vec{0}$ get contradiction:

If all zero then ρ contained in



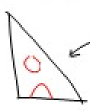
say in this triangle

But all loops in the disk are iness.

Exercise: Can isotope ρ to arrange

$$\begin{pmatrix} |\lambda\cap\rho| \\ |\mu\cap\rho| \\ |\nu\cap\rho| \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$ simultaneously.

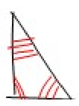
Now:



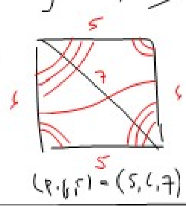
not allowed as arcs of $\rho \cap T$ or T'

[T, T' are my triangles]

Allowed

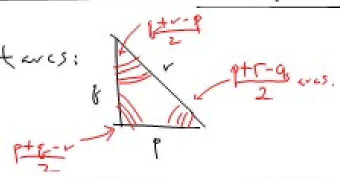


So we see

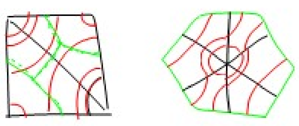


$(p,q,r)=(5,6,7)$

Lets count arcs:



Cut and paste



Very Bad!: ρ not connected and has trivial loop

Deduce: either

$p+q=r$ ~ (*)

$q+r=p$ ~

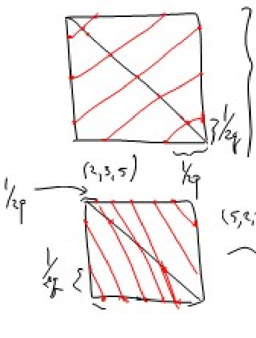
$r+p=q$.

Exercise: ρ connected iff $\gcd(p,q)=1$.

[Generally $\gcd(p,q)$ counts the # of components]

Assuming one of (*) holds.

Now: Isotope the points of $\rho\cap\lambda$ so that they are evenly spaced, similarly for $\rho\cap\mu$, $\rho\cap\nu$



Now relative to these intersections isotope ρ to be straight.

Claim: If $p+q=r$ then ρ is the line of slope p/q.

$\rightsquigarrow$ if $p=q+r$ then find slope $-p/q$ (and similarly for $q=p+r$)

Thus: $\mathcal{S}(\mathbb{T}^2) \cong \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$

i.e. the set of slopes.

Exercise: if α has slope $\frac{p}{q}$, β has slope $\frac{r}{s}$

then $i(\alpha, \beta) = \left| \det \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right|$

i.e. geom. int # = | algebraic int # |

To prove $\mathcal{S}(\mathbb{T}) = \widehat{\mathbb{Q}}$ we made choices.

Choices are bad!

Avoid making "special choices" as follows:

Instead of mapping $\mathcal{S}(\mathbb{T}^2) \longrightarrow \mathbb{R}_{\geq 0}^{\{\lambda, \mu, \nu\}}$

replace with the map $\mathcal{S}(\mathbb{T}) \xrightarrow{\ i\ } \mathbb{R}_{\geq 0}^{\mathcal{S}(\mathbb{T})}$

$$\mathbb{R}_{\geq 0}^{\mathcal{S}(\mathbb{T})} = \{ f : \mathcal{S}(\mathbb{T}) \to \mathbb{R}_{\geq 0} \}$$

$$\alpha \longmapsto \big(\beta \mapsto i(\alpha, \beta) \big) = \sigma_\alpha \in \mathbb{R}_{\geq 0}^{\mathcal{S}(\mathbb{T})}$$

We give $\mathbb{R}_{\geq 0}^{\mathcal{S}}$ the topology of point-wise convergence.

Extend i to a function:

$$i : \mathbb{R}_{\geq 0} \times \mathcal{S} \longrightarrow \mathbb{R}_{\geq 0}^{\mathcal{S}}$$
$$t \cdot \alpha \longmapsto t\sigma_\alpha$$

So: $i(\mathbb{R}_{\geq 0} \times \mathcal{S})$ is a countable union of rays.

Def Let $\mathcal{ML}(S) = \overline{i(\mathbb{R}_{\geq 0} \times \mathcal{S}(S))}$

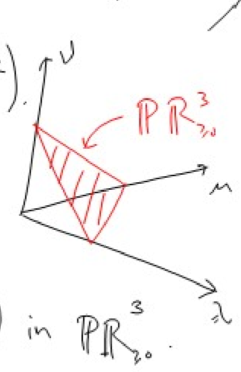
= the space of measured laminations

Def: If V has an $\mathbb{R}_{>0}$ action

Let $\mathbb{P}V = \mathbb{P}(V - \{0\}) = {}^{V - \{0\}} / _{v \sim t \cdot v}$

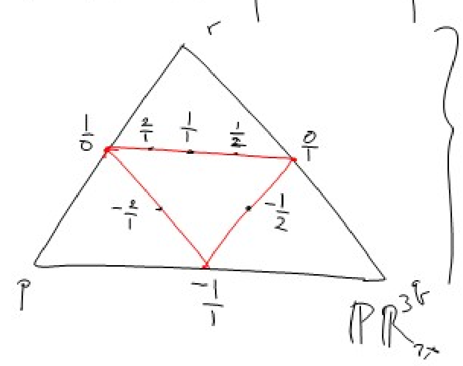
Lets draw $\mathbb{P}\mathcal{ML}(\mathbb{T}^2)$.

Consider $\mathbb{R}_{\geq 0}^{\{\lambda, \mu, \nu\}}$



β gives coordinates (p, q, r) in $\mathbb{P}\mathbb{R}^3_{\geq 0}$.

Condition (*) implies that β lies on one of 3 lines:



$\mathcal{S}$ fills up the circle like $\widehat{\mathbb{Q}}$ fills the circle

Exercise: Prove that $\mathbb{P}\mathcal{ML}(\mathbb{T}) \cong S^1$ //