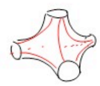
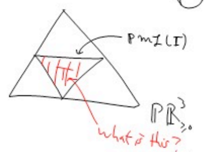
<u>Exercises</u>: $\mathcal{S}(S_{1,1}) \cong \hat{\mathbb{Q}}$, $\mathcal{S}(S_{0,4}) \cong \hat{\mathbb{Q}}$

[Hint: use arcs instead.  Tetrahedron.]

<u>Last time</u>:



<u>Recall</u>: If $\alpha \in \mathcal{S}(\mathbb{T})$ then $\sigma_\alpha \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$.
$\sigma_\alpha(\beta) = i(\alpha, \beta)$. Also $\sigma_{t\alpha} = t \cdot \sigma_\alpha$.

<u>Answer to question</u>: T is the set of (p, q, r)
so that
$$\left.\begin{array}{l} p+q<r \\ q+r<p \\ r+p<q \end{array}\right\} \text{ triangle inequalities.}$$

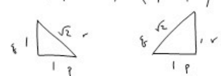
<u>That is</u>: T is the space of <u>marked</u> triangles in $\mathbb{E}^2$, up to isometry and scaling

<u>Eg</u>: $(1,1,1) \rightsquigarrow$  equilat.

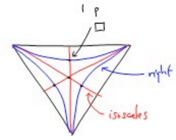
$(p,p,r) \rightsquigarrow$  isosceles

$(p, q, \sqrt{p^2+q^2}) \rightsquigarrow$  right.

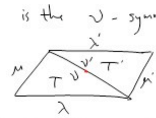
<u>Oops</u>: $(1,1,\sqrt{2}) \neq (1,\sqrt{2},1)$



<u>Here picture</u>:



<u>Tori</u>: If we have T a triangle then there is the ν-symmetric copy T'



} rotate about midpoint.

Glue $\lambda \to \lambda'$, $\mu \to \mu'$ to get a torus.

So get all <u>flat tori</u>.

<u>Def</u>: $\mathrm{Teich}(\mathbb{T}^2) = \left\{ (f, X) \;\middle|\; \begin{array}{l} f: \mathbb{T}^2 \to X \text{ is homeo} \\ X \text{ is flat} \end{array} \right\} \Big/ \simeq$

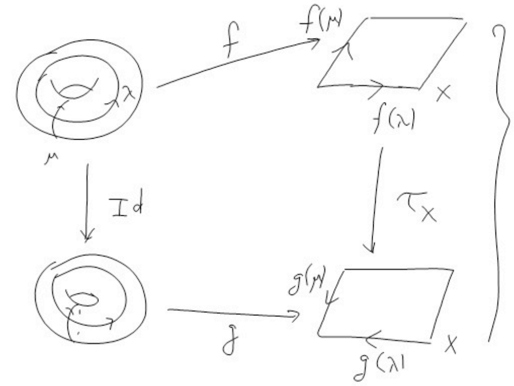
$(f, X) \simeq (g, Y)$ iff there is $h \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and a <u>similarity</u> $h': X \to Y$ so that

$$\begin{tikzcd} \mathbb{T}^2 \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h'"] \\ \mathbb{T}^2 \arrow[r, "g"] & Y \end{tikzcd} \quad \Bigg\} \text{ commutes.}$$

<u>Rmk</u>: Every X has a symmetry (isometry)

} call this the <u>elliptic involution</u>, τ_X

[Why elliptic? An elliptic equation is one of the form $y^2 = P(x)$, P has degree 3
This defines an elliptic curve in $\mathbb{C}^2$ (really in $P\mathbb{C}^2 = \mathbb{P}(\mathbb{C}^3)$). <u>Claim</u>: elliptic curves admit an involution. If (x, y) is a solution so is $(x, -y)$. <u>Note</u>: $y \in \mathbb{C}$ so $y \to -y$ is 180° rotation.]



Commutes So $(f,X) \simeq (g,X)$ in Teich$(\mathbb{T})$

"The elliptic involution acts trivially on Teich$(\mathbb{T}^2)$."

Def: Suppose $(f,X) \in$ Teich$(\mathbb{T})$. Define

$$\ell_{(f,X)}(\beta) = \ell_X(\beta) = \mathbb{E}^2 \text{ length of the geodesic rep of } \beta.$$

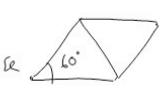
[Here X is a flat torus: If we scale X then ℓ_X is also scaled] So get map

$$\ell : \text{Teich}(\mathbb{T}) \longrightarrow \mathbb{P}\mathbb{R}^{\mathcal{S}}_{\geq 0}.$$

Ex: 

$\ell_{\square}\left(\frac{-1}{1}\right) = \sqrt{2}$.

Easy: $\ell_{\square}\left(\frac{p}{q}\right) = \sqrt{p^2+q^2}$

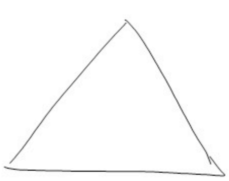
Easy: Compute $\ell_{\hexagon}\left(\frac{p}{q}\right)$. Here use 

Exercise: Given X compute $\ell_X\left(\frac{p}{q}\right)$

"Clearly, Teich$(\mathbb{T})$ is homeo to $\mathcal{T}$, the space of marked triangles." [Work]

So: Teich$(\mathbb{T})$ is an open disk and

Teich$(\mathbb{T}) \cup \mathcal{PML}(\mathbb{T})$ is a closed disk



$\mathbb{P}\mathbb{R}^3_{\geq 0}$