Finish last time: Suppose $X_i \in \text{Teich}(T^2)$

and $\sigma \in PML(T^2)$ so that
$$l_{X_i}\begin{pmatrix}0\\ \infty\\ -1\end{pmatrix} \longrightarrow \sigma\begin{pmatrix}0\\ \infty\\ -1\end{pmatrix}$$
as $i \to \infty$. Prove that $l_{X_i} \to \sigma$ as $i \to \infty$

Consequence: If $X_i \to \sigma_{p/q}$ then $l_{X_i}(p/q) \to 0$

Prop: $MCG(T) \cong GL(2,\mathbb{Z})$.

Pf: **Step 1:** Recall that $\mathbb{Z}^2 \to \mathbb{R}^2 \to T^2$
$\circlearrowleft$
$GL(2,\mathbb{Z})$ acts

Check that every $A \in GL(2,\mathbb{Z})$ descends to a homeomorphism f_A. I.e. map
$$\begin{array}{ccc} GL(2,\mathbb{Z}) & \longrightarrow & \text{Homeo}(T^2) \\ \Psi & & \Psi \\ A & \longmapsto & f_A \end{array}$$

Step 2: The map $GL(2,\mathbb{Z}) \longrightarrow MCG(T)$ is injective.

[Use $\mathcal{S}(T^2)$ as isotopy invariants: Eg if $f \simeq g$ then f, g induce the same permutation on $\mathcal{S}(T^2)$]

[I.e. $f \simeq Id \Rightarrow f$ preserves all slopes]

So: suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $f_A \simeq Id$

So: $f_A(p/q) = p/q$ However:

Exercise: $f_A\left(\frac{p}{q}\right)$ has slope $\frac{dp+cq}{bp+aq}$.

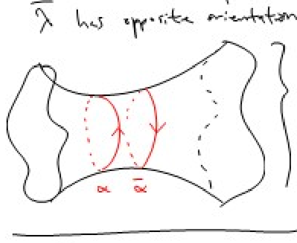
Thus if $f_A = Id \in MCG(T)$ find

$$\left.\begin{aligned} f_A\left(\tfrac{0}{1}\right) &= \tfrac{c}{a} = \tfrac{0}{1} \\ f_A\left(\tfrac{1}{0}\right) &= \tfrac{d}{b} = \tfrac{1}{0} \\ f_A\left(\tfrac{-1}{1}\right) &= \tfrac{-d+c}{-b+a} = \tfrac{-1}{1} \\ &= \tfrac{1}{-1} \Rightarrow -b+a = d-c \end{aligned}\right\} \quad \left.\begin{aligned} &c = 0 \\ &b = 0 \\ &\cancel{b}^{\,0} - a = -d + \cancel{c}^{\,0} \end{aligned}\right\} \quad \begin{aligned} &A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \\ &\text{implies that} \\ &a = d = \pm 1 \end{aligned}$$

ok fine.

2 possibilities $A = \pm Id$.

Use oriented loops. If λ is oriented then $\bar{\lambda}$ has opposite orientation.

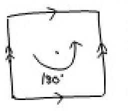


$\alpha, \bar{\alpha}$ are not isotopic (unless $S = S^2$, two-sphere]

So: $f = f_{-Id}$ then

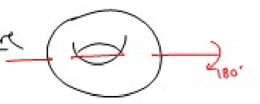
f reverses all loops. // injective.

[Notice: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts on T^2 via $180°$ rotation:

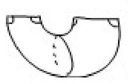


f_{-Id} has 4 fixed pts; This is another rep'n of the elliptic elt.

Picture

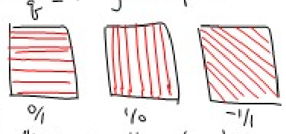


↓ mod out to get



$S^2(2,2,2,2) = S^2$ with 4 marked points

Exercise: Every slope $\frac{p}{q} \subseteq T^2$ gives a "product structure" on T^2



Check this product is invariant under the elliptic. Use this to show (again) that $\mathcal{S}(S_{0,4}) \simeq \hat{\mathbb{Q}}$.

Step 3 Surjective. We use the Alexander method

Proposition: Suppose $f \in \text{Homeo}^+(S, \partial)$. Suppose that $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ is a collection of loops and arcs that
i) $\alpha_i \cup \alpha_j$ is tight $\forall i,j$
ii) $\mathcal{A}$ fills S
iii) $f(\alpha_i) \simeq \alpha_i \ \forall i$ ⊛

$\Longrightarrow f \simeq Id$, keeping ∂S fixed throughout.

⊛ As oriented loops/arcs.

Def: $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ fills if the α_i are tight and $S \setminus \mathcal{A}$ is a collection of disks.

Pf of Alex Method: Post compose f with $g \in \mathrm{Homeo}_0(S, \partial)$

s.t. $g(f(\alpha)) = \alpha$, i.e. $g \circ f|_\alpha = Id_\alpha$.

Note $g \circ f \simeq f$. Now cut. Let $S' = S_\alpha = S \setminus \mathring{N}(\alpha)$.

Check that $f' \equiv f|S'$ again satisfies hypotheses

with $a' = a|S'$. Now induct, Base case is Alex Trick //

Exercise: Induct on what?

Now for step 3: Fix any $f \in MCG(T^2)$.

Let $\alpha = f\binom{0}{1}$, $\beta = f\binom{1}{0}$. so $i(\alpha, \beta) = 1$

So the slopes p/q, r/s for α, β satisfy

$\det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1$. Let $A = \begin{pmatrix} -r & s \\ p & -q \end{pmatrix}$ and

verify that $f_A \circ f$ fixes the slopes $\frac{0}{1}, \frac{1}{0}$.

Notice that there is a $B \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

s.t. $f_B \circ f_A \circ f$ fixes $\frac{0}{1}, \frac{1}{0}$ as oriented curves,

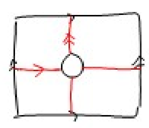
Apply Alex Method to see $f_B \circ f_A \circ f \simeq Id$

So $f \simeq (f_B \circ f_A)^{-1}$. Thus $f \simeq f_{(BA)^{-1}}$ //

[last step — untrustworthy! check it yourself.]

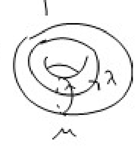
Exercise: $MCG(S_{1,1}) \simeq GL(2, \mathbb{Z})$

use arcs as shown:



Challenge: Compute $MCG(S_{0,4})$ using these tools.

Computation:



$T_\lambda \cong f_{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}$

$T_\mu \cong f_{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$

[$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$]

Braid relation: $T_\lambda T_\mu T_\lambda \cong T_\mu T_\lambda T_\mu$.

This follows directly from $ABA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = BAB$

But there is a better proof!

[Think about: Why does A have a -1 in it???]