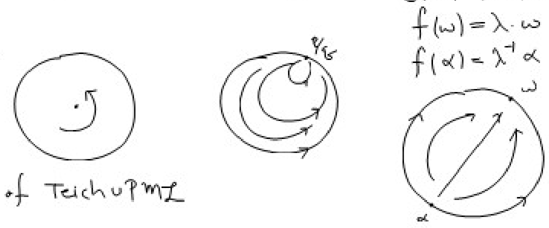
Classify elements in $SL(2,\mathbb{Z})$: A, f_A

Notation: elliptic → "hyperelliptic"

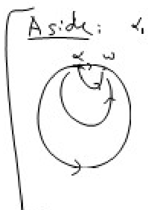
Elliptic	Parabolic	Hyperbolic
$\lvert \operatorname{trace} A \rvert < 2$	$\lvert \operatorname{trace} A \rvert = 2$	$\lvert \operatorname{trace} A \rvert > 2$
finite order [periodic, torsion] $\exists k$ s.t. $f_A^k \simeq f_{A^k} \simeq \mathrm{Id}$	Power of a Dehn twist $f_A = T_{\varphi_\gamma}^k$	Anosov
fixes a point in Teich	fixeds φ_γ in $\mathcal{PML}$	fixes $\alpha, \omega \in \mathcal{ML}$, $\exists \lambda > 1$ so that $f(\omega) = \lambda \cdot \omega$, $f(\alpha) = \lambda^{-1} \alpha$

Pictures

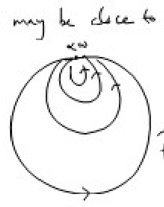


of Teich ∪ $\mathcal{PML}$

Aside: α, ω after conjugation may be close to each other



or really close



$\underset{\text{to}}{\overset{\text{limit}}{\rightsquigarrow}}$ parabolic.

arise as symmetries of some flat torus.	After isotopy fix a loop	fix pair of filling laminations

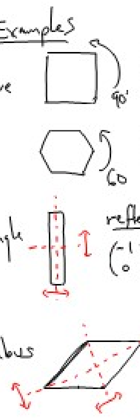
Examples

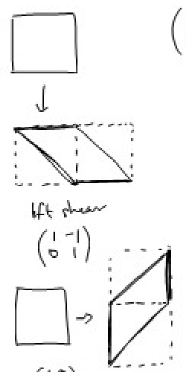
square $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

hex $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

rectangle — reflections $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

rambus $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$





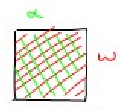
left shear $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ again left

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2$

Crazy!



α, ω are foliations in the eigen-directions.

Suppose $S \neq \mathbb{T}^2$ and $\chi(S) < 0$
$\neq S_{0,3}$

Then we define $\mathcal{PML}(S)$ exactly as before:

$\mathbb{R}_{\geq 0} \times \mathcal{S} \xrightarrow{i} \mathbb{R}_{\geq 0}^{\mathcal{S}}$. $\mathcal{ML}(S)$ is the closure, $\mathcal{PML}$ is the projectivization. Def: $\xi(S_{g,b}) = 3g-3+b$

↖ \xi = ξ

[Thurston] $\mathcal{PML}(S) \cong S^{2\xi(S)-1}$.

Def: $\operatorname{Teich}(S) = \left\{ (f, X) \,\middle|\, \begin{array}{l} f: \mathring{S} \to X,\ f \text{ homeo}, \\ X \text{ fin. volume hyperbolic surface} \end{array} \right\} \Big/ (f,X) \simeq (g,Y)$

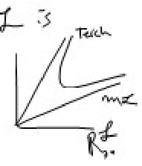
$(f,X) \sim (g,Y)$ iff

$$\left.\begin{array}{ccc} \mathring{S} & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & \circ & \downarrow{\scriptstyle h'} \\ \mathring{S} & \xrightarrow{g} & Y \end{array}\right\} \begin{array}{l} h \in \operatorname{Homeo}_0(S) \\ h' \text{ is an isometry.} \end{array}$$

Thm: $\operatorname{Teich}(S) \simeq \mathring{\mathbb{B}}^{2\xi(S)} = \mathring{\mathbb{B}}^{6g-6+2b}$.

[Thurston] $\mathcal{PML}$ is ~~the~~ set of accumulation pts of $\operatorname{Teich} \hookrightarrow \mathbb{P}\mathbb{R}_{\geq 0}^{\mathcal{S}(S)}$. Teich ∪ $\mathcal{PML}$ is a closed ball.

Picture



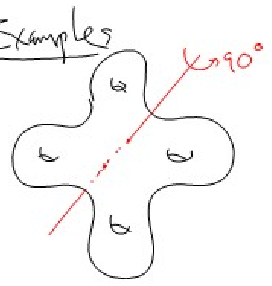
Nielsen–Thurston Classification

$f \in \mathrm{MCG}(S)$

periodic	reducible	pseudo-Anosov.
$\exists k$ s.t. $f^k = \mathrm{Id}$	$\exists$ a rational point in $\mathcal{PML}$ fixed by f i.e. f fixes a multicurve	$\exists \alpha, \omega \in \mathcal{ML}(S)$ both filling s.t. $f(\alpha) = \frac{1}{\lambda}\alpha$, $f(\omega) = \lambda\omega$, $\lambda > 1$.

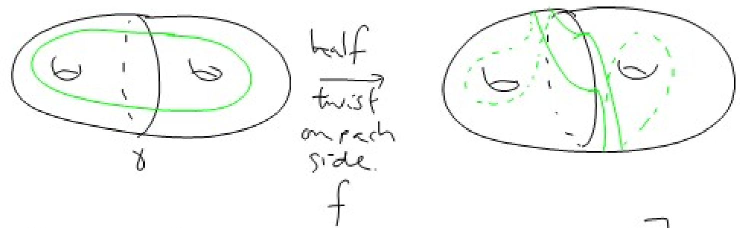
Def: $\omega \in \mathcal{ML}(S)$ is filling if $i(\gamma, \omega) \neq 0$ $\forall \gamma \in \mathcal{S}(S)$.

Examples



Exercise: Find a elliptic element in $MCG(S)$ which is not induced by $S \hookrightarrow \mathbb{R}^3$ ↺ symmetry.

Reducible:



Notice: $f^2 \simeq T_\gamma^2$ [a nonstandard root!]

Thm: Every element of $MCG(S)$ is either periodic, reducible, or pseudo Anosov

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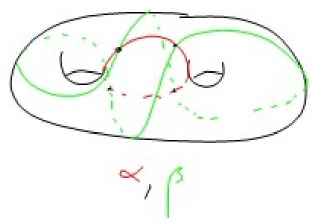
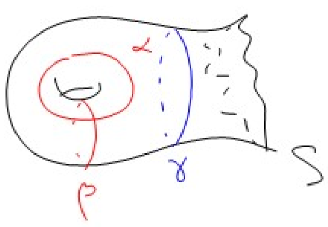
Ex:



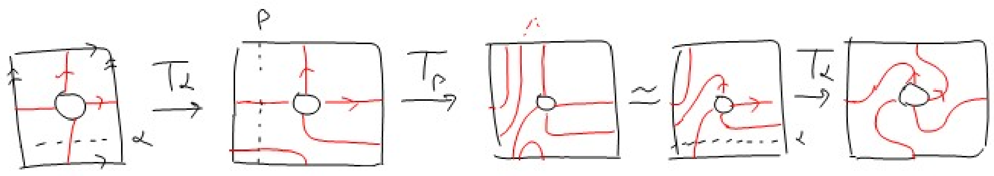
Def: $f = T_\alpha \circ T_\beta^{-1}$

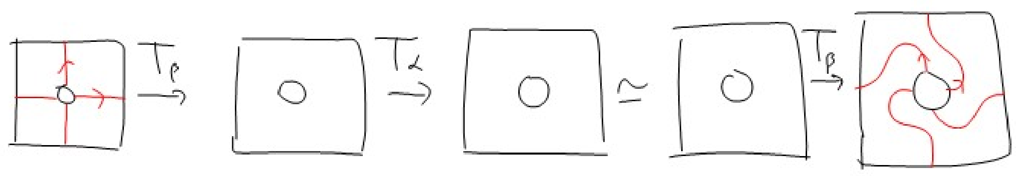
Thurston/Veech construction: f is pA.

Braid relation:



γ cuts a handle off of S.





Exercise: Fill in the second line and check that

$$T_\alpha T_\beta T_\alpha \simeq T_\beta T_\alpha T_\beta.$$

Also: $T_\alpha T_\beta T_\alpha$ is "$\frac{1}{4}$ of a Dehn twist about γ".

Also: $\left.\begin{array}{l} T_\alpha T_\beta T_\alpha(\alpha) = \beta \\ T_\alpha T_\beta T_\alpha(\beta) = \bar{\alpha} \end{array}\right\}$ as oriented loops

Also: $(T_\alpha T_\beta T_\alpha)^2 = T_\alpha T_\beta T_\alpha \underbrace{T_\beta T_\alpha}_{\text{braid}} T_\beta = (T_\alpha T_\beta)^3$

So: $(T_\alpha T_\beta T_\alpha)^4 = (T_\alpha T_\beta)^6 = T_\gamma$.

Rmk: $(T_\alpha T_\beta)^3 \simeq$ hyperelliptic in $\mathbb{T}^2$

So: Twists can permute/reverse loops.