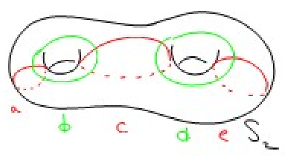
Exercise: Consider the 5 loops a, b, c, d, e



Suppose $x, y \in \{a, b, c, d, e\}$

Shorthand: use x to denote T_x.

(i) $[x, y] = 1$ if $i(x, y) = 0$ [for commutativity]

(ii) $xyx = yxy$ if $i(x, y) = 1$ [braid relation]

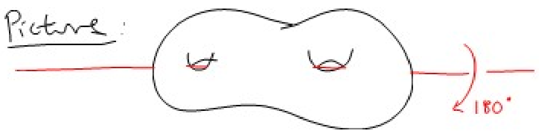
(iii) $(abcde)^6 = 1$ [chain relation]

Notation: $\Delta = abcdeedcba$

(iv) $\Delta^2 = 1$ [hyperelliptic]

(v) $[\Delta, x] = 1$ [hyperelliptic is central]

Picture:



Exercise: Check the relations above. Check that $\Delta = 180^\circ$ rotation shown.

[**Rmk**: In fact $\langle a, b, c, d, e \mid \text{relations} \rangle$ is a presentation of $MCG^+(S_2)$ [Birman–Hilden]]

Dehn–Lickorish.

① $MCG(S)$ is generated by a finite collection of twists and half twists and one reflection.

② $MCG^+(S, \partial)$ is generated by a finite collection of twists.

② $\Longrightarrow$ ①

$$\begin{array}{ccccc} & & & & MCG(S) \\ & & & & \triangledown \\ \mathbb{Z}^{|\partial S|} & \longrightarrow & MCG^+(S, \partial) & \twoheadrightarrow & MCG^+_\partial(S) \end{array}$$

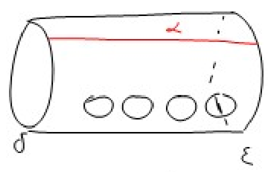
} finite index $MCG(S)$ is gen by elts of $MCG^+_\partial(S)$ and a reflection, and enough half twists to permute ∂S

The kernel is generated by Dehn twists about ∂S.

So ② $\Rightarrow$ ①. //

Case 1 $g = 0$, $b \geq 5$ [$b = 4$ was an exercise.]

Fix components $\delta, \varepsilon \subset \partial S$. Fix an arc α connecting δ to ε.

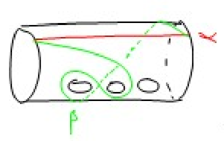


Notice: $MCG^+(S, \partial \cup \alpha) \cong MCG^+(S_\alpha, \partial)$ By induction, this is fin. gen by twists.

Fix $f \in MCG^+(S, \partial)$. Let $\beta = f(\alpha)$ {Suppose that α, β are tight.

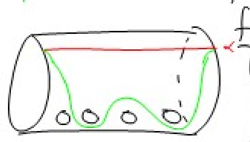
Case 1a: If $\beta \simeq \alpha$ we are done, by induction.

Case 1b: Suppose that $i(\alpha, \beta) = 2$



} Apply T_δ, T_ε and $g \in MCG^+(S_\alpha, \partial)$ to put β in the following form:

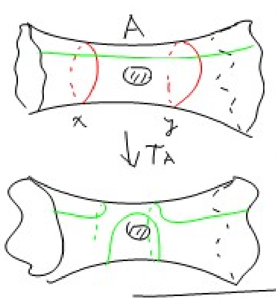
Now



} So $\alpha \cup \beta$ partitions $\partial S \setminus (\delta \cup \varepsilon)$ in simplest possible fashion.

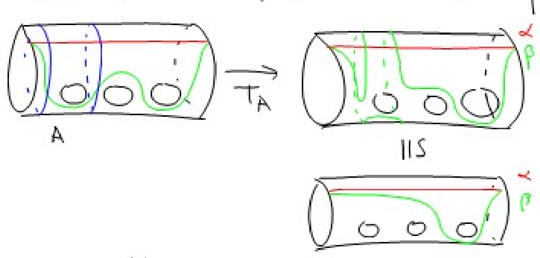
This is possible by the classification of multiarcs in S. [Classification of surfaces].

Double Dehn twists : (Point-Pushing).



$T_A = T_x \cdot T_y^{-1}$ is the double twist

So: If α, β meet only at $\partial\alpha = \partial\beta$ then, After applying $T_\delta, T_c, g \in MCG^+(S_\alpha, \partial)$ and some double Dehn twists, $\alpha \simeq \beta$



So Case 1b is done.

Case 1c $\quad i(\alpha, \beta) \geq 3$.

Claim: [Surgery] There $\exists$ a sequence of arcs $\{\alpha_i\}_{i=0}^{N}$ s.t.

(i) $\alpha_0 = \alpha, \ \alpha_N = \beta$

(ii) $\partial\alpha_i = \partial\alpha \quad \forall i$

(iii) $i(\alpha_i, \alpha_{i+1}) = 2 \quad \forall i = 0, 1, \ldots, N-1$

[α_i, α_{i+1} fall into case 1b.]

Claim $\Rightarrow$ we are done:

Pf: Let f_0 be the class $f_0(\alpha_1) \cong \alpha_0$

Let f_1 " " " $f_1 f_0(\alpha_2) \simeq \alpha_0$

Generally f_i has $f_i f_{i-1} \cdots f_0(\alpha_{i+1}) \simeq \alpha_0$

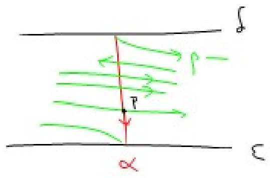
So: $f_{N-1} f_{N-2} \cdots f_0(\alpha_N) = \alpha$ i.e.

$f_{N-1} f_{N-2} \cdots f_0(f(\alpha)) = \alpha$. [Recall $f(\alpha) = \beta$]

So: $f_{N-1} \cdots f_0 \cdot f \in MCG^+(S_\alpha, \partial)$

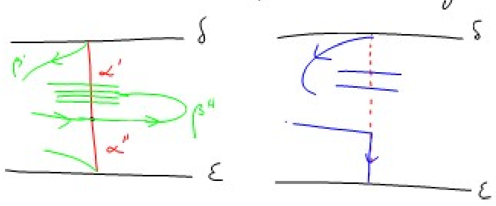
and we are done, by induction. //

Pf of Claim:



p is the last point of $\alpha \cap \beta$, along α.

Let $\alpha = \alpha'' \cup_p \alpha'$, $\beta = \beta'' \cup_p \beta'$ } α'' meets p and ε; α' " p " δ [Similarly for β]



surger

Let $\beta_1 = \beta' \cup \alpha''$

[$\beta_1 = \alpha_{N-1}$]

Observe: $\beta_1 \cap \beta = \partial\beta$ and, $|\beta_1 \cap \alpha| + 1 \leq |\beta \cap \alpha|$

Now, induction on $\alpha \cap \beta$ proves the claim. //

This finishes the proof of DL Theorem. [$g = 0$].

Picture: