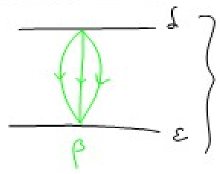
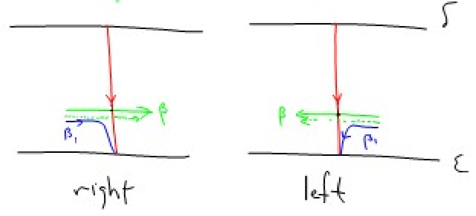
Pictures to clarify "push-offs"

Picture:



Can isotope β (rel ∂) to the left or right

Cases



Thus, when we surger, to obtain β_1, we find

$\beta_1 \cap \beta = \partial \beta$ and $i(\beta_1, \alpha) \leq |\beta_1 \cap \alpha| < i(\beta, \alpha)$.

Before Case 2 $(g > 0)$ we need:

Def: If $\alpha, \beta \subset S$ are nonsep. and $i(\alpha, \beta) = 1$ Say α, β are dual [Check: If $i(\alpha, \beta) = 1$ $\Rightarrow$ both α, β are nonsep.]

Def: If α, β are nonsep loops then there is a finite sequence $\{\alpha_i\}_{i=0}^{N}$ of loops

(i) $\alpha_0 = \alpha$, $\alpha_N = \beta$

(ii) α_i is dual to α_{i+1} $\forall i \in \{0, 1, 2, \cdots N-1\}$

Call $\{\alpha_i\}$ a chain [Abuse of notation]
[length of chain = N]

[Rmk: arcs connecting distinct ∂-components of S and nonsep loops have lots in common!]

Lemma [Surgery]: $\forall \alpha, \beta$ non sep. $\exists$ a chain connecting α to β.

Pf: Induct on $i(\alpha, \beta)$. [Will show the length is at most $2i(\alpha, \beta) + 2$]

If $\alpha = \beta$ then length is zero.

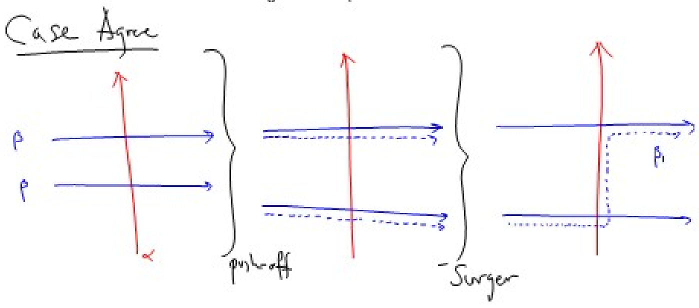
If α, β dual " " " one.

If $i(\alpha, \beta) = 0$ then there is a chain of length 0 or 2.
I.e. $\exists \alpha_1$ dual to both α, β. [Exercise]

[Very similar to property (vi) of $i(\cdot, \cdot)$]

Now: Suppose that $i(\alpha, \beta) \geq 2$. $\alpha \cup \beta$ tight
Pick $x, y \in \alpha \cap \beta$. conseq. on α

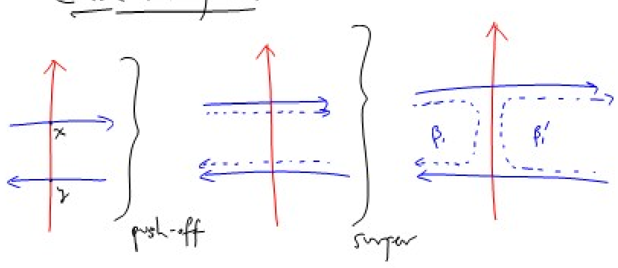
Case Agree



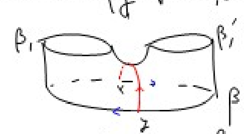
Notice: $i(\beta_1, \alpha) \leq |\beta_1 \cap \alpha| < i(\beta, \alpha)$ [we've lost x]

Also: β_1, β meet once, transversely $\Rightarrow$
$i(\beta_1, \beta) = 1 \Rightarrow \beta_1, \beta$ are dual.

Case Disagree



Notice: β, β_1, β_1' cobound a copy of $S_{0,3}$.



[Easy Exercise: β non sep $\Rightarrow$ at least one of β_1, β_1' also nonsep.]

WLOG: (i) β_1 nonsep. Also, (ii) $\beta \cap \beta_1 = \emptyset$.

(iii) $i(\beta_1', \alpha) + i(\beta_1, \alpha) \leq |\beta_1' \cap \alpha| + |\beta_1 \cap \alpha| < i(\beta, \alpha)$.

In both cases: α is connected to β_1 by a chain, and β_1 connected to β by a chain of length ≤ 2. // Surgery.

Case 2 $(g>0)$ of the Pf of Lickorish Thm.

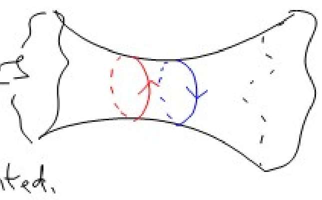
Fix $\alpha, \gamma \subset S$ dual nonsep loops. Fix $f \in$ $\in \mathcal{MCG}^+(S,\partial)$. Let $\beta = f(\alpha)$.

Case 2a: $\beta \simeq \alpha$

Case 2a(i): $\alpha \simeq \beta$ as oriented loops

Thus $f \in \mathcal{MCG}^+(S, \partial \cup \alpha) \twoheadleftarrow \underset{[\text{Exercise}]}{} \mathcal{MCG}^+(S_\alpha, \partial)$ and the latter is fin gen, by twists, by induction on $g = \text{genus}(S)$.

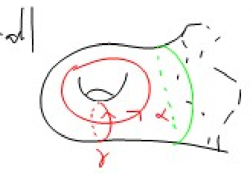
Case 2a(ii): Picture



α, β oppositely oriented.

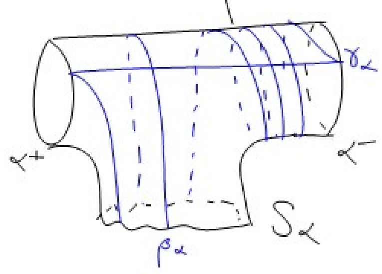
So: $(T_\alpha T_\gamma)^3 \circ f(\alpha) \simeq \alpha$ as oriented curves.

[Recall



$(T_\alpha T_\gamma)^3$ in $N(\alpha \cup \gamma)$ is the hyperelliptic]

Case 2b: α, β are dual. Isotope to arrange $\alpha \cap \gamma = \alpha \cap \beta$. Let $\gamma_\alpha = \gamma \cap S_\alpha$, $\beta_\alpha = \beta \cap S_\alpha$



By the topological classif. of arcs $\exists g \in \mathcal{MCG}^+(S_\alpha, \partial)$ s.t. $g(\beta_\alpha) = \gamma_\alpha$.

Thus $g(\beta) = \gamma$, and $T_\alpha T_\gamma T_\alpha \, g f(\alpha) \simeq \alpha$ and we are in case 2a.

Case 2c: Suppose that $\{\alpha_i\}$ is a chain of length ≥ 2 connecting α to $\beta = f(\alpha)$.

$\forall i \; \exists f_i \in \langle T_\alpha, T_\gamma, \mathcal{MCG}^+(S, \partial \cup \alpha) \rangle$ s.t. $f_i(f_{i-1} f_{i-2} \cdot \dots \cdot f_1(\alpha_i)) \simeq \alpha_0$ by case 2b.

[Think: $f_1(\alpha_1) \simeq \alpha_0$ by case 2b.
Now: $i(f_1(\alpha_2), f_1(\alpha_1)) = i(\alpha_2, \alpha_1) = 1$
So $f_1(\alpha_2)$ is dual to $f_1(\alpha_1) \simeq \alpha_0$ and $\exists f_2$ s.t. $f_2(f_1(\alpha_2)) \simeq \alpha_0$, etc.]

We find that

$$f_N \cdot f_{N-1} \cdot \dots \cdot f_1(\alpha_N) \simeq \alpha_0$$

As $\alpha_N = \beta = f(\alpha)$ we are done. [we are in Case 2a]

// Lickorish Thm