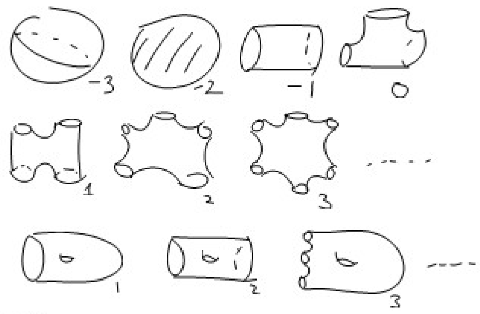
Recall: $S = S_{g,b}$ is the connected, compact, orientable surface with $\text{genus}(S) = g$, $|\partial S| = b$.

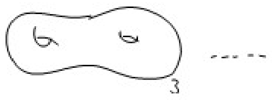
Picture: 

if $b=0$ write $S = S_g$

Def: $\xi(S) = 3g - 3 + b$ is the complexity of S

[Note $\chi(S) = 2g - 2 + b$, is the Euler char.]

Picture 



$S, \xi(S)$.

Recall: $\mathcal{S}(S) = \{\text{ess, nonperip. loops in } S\}/_{\text{isotopy}}$

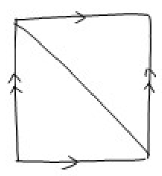
Def [Harvey] Suppose that $\xi(S) \geq 2$.

$\mathcal{C}(S)$, the complex of curves, is the simplical complex with vertex set equal to $\mathcal{S}(S)$ and $\{\alpha_0, \alpha_1, \dots, \alpha_k\}$ is a simplex if $i(\alpha_i, \alpha_j) = 0$ $\forall i, j$.

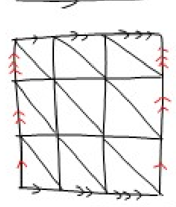
Recall: A simplical complex has a vertex set V and a collection of simplices $\mathcal{C} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ so that

(*) if $\emptyset \neq \tau \subseteq \sigma \in \mathcal{C}$ then $\tau \in \mathcal{C}$.

[i.e. $\mathcal{C}$ is closed under the operation of taking non-empty subsets]

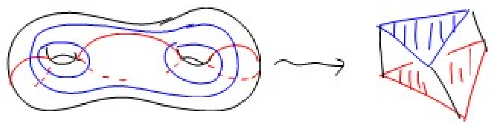
Not Ex: 

is not a simplicial complex

Ex: 

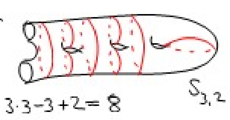
has 9 vertices, etc
every simplex now determined by its vertices.

Picture of a bit of $\mathcal{C}(S_2)$



Exercises: (i) Every simplex in $\mathcal{C}(S)$ is a facet of a simplex with $\xi(S)$ many vertices.

(ii) The maximal dimension of any simplex is $\xi(S) - 1$.

(iii) Def: The vertizes of a max'l simplex form a pants decomposition 

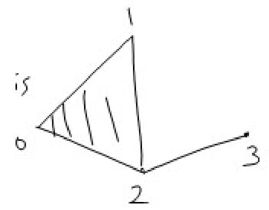
$3 \cdot 3 - 3 + 2 = 8$

(iv) List all pants decomp. of all surfaces with complexity ≤ 3, up to homeomorphism.

Exercise: The underlying topological space $|\mathcal{C}(S)|$ is connected. [Hint: use the surgery lemmas from last week.]

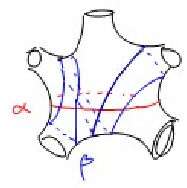
[Recall: If $X \subseteq \mathcal{P}(V)$ is a simplical complex then
$|X| = \bigcup_{\sigma \in X} \Delta^{|\sigma|} / \sim$ $x \sim y$ if x, y have the same "barycentric coords"]

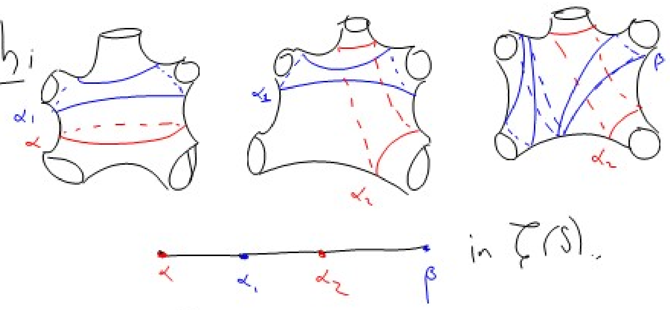
Ex: $V=\{0,1,2,3\}$, $X=\{\{0,1,2\},\{2,3\}\}$ and close under taking nonempty subsets.

$|X|$ is 

Def: Distance in $\mathcal{C}(S)$. For any $\alpha,\beta\in\mathcal{S}(S)$ define

$$d_S(\alpha,\beta)=\min\left\{n \;\middle|\; \begin{array}{l} n \text{ is the length of some} \\ \text{edge path connecting } \alpha,\beta \\ \text{in } \mathcal{C}^1(S)\end{array}\right\}$$

Ex:  $\rightsquigarrow$ This gives a lower bound on the diameter of $\mathcal{C}(S_{0,5})$

Path:  in $\mathcal{C}(S)$.

$\therefore\ d_S(\alpha,\beta)\leq 3.$

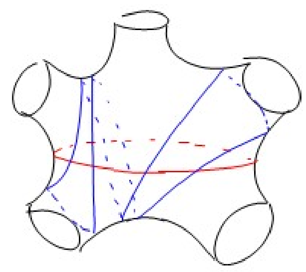
Def: Suppose $\alpha,\beta\in\mathcal{S}(S)$ and $\alpha\cup\beta$ tight and $S\setminus(\alpha\cup\beta)$ is a collection of disks and peripheral annuli. Then say α,β fill S.

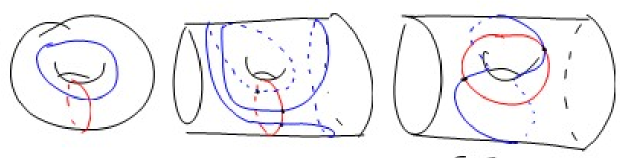
Exercise: α,β fill

iff $d_S(\alpha,\beta)\geq 3$

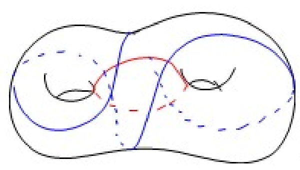
iff $\forall\gamma\in\mathcal{S}(S)$, $i(\alpha,\gamma)\neq 0$ or $i(\beta,\gamma)\neq 0$.

[Hint: Picture of $d_S=2$ ]

So:  α,β fill so $d_S(\alpha,\beta)\geq 3$. So $d_S(\alpha,\beta)=3$.

Examples: 

[?]

 [Casson – Bleiler].

Exercise: Compute distances.

Lemma: If $\xi(S)\geq 1$ (or $S=T^2$) then $\forall\alpha\ \exists\beta$ s.t. α,β fill.

Corollary: If $\xi(S)\geq 2$ then $\mathrm{diam}(\mathcal{C}(S))\geq 3$.

Next time: Ivanov's Thm.