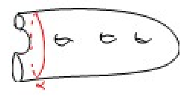
**Recall:** If $X$ is a simplical complex and $\sigma \in X$ then $lk(\sigma) = \{\tau \in X \mid \sigma \cup \tau \in X, \sigma \cap \tau = \emptyset\}$.

**Picture:**




if $\sigma = \{\alpha\}$ then $lk(\sigma) \cong \mathcal{C}(T_\alpha)$

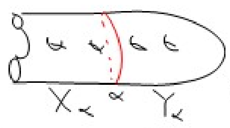
$S_\alpha = P_\alpha \cup T_\alpha$. $P_\alpha$ is a pants, $T_\alpha$ is the other component.

**Picture:**


Here $lk(\alpha) \cong \mathcal{C}(S_\alpha)$ if $\alpha$ nonsep.
[Slight abuse of Notation]

**Notice:** $\mathcal{C}(S)$ is locally infinite.

**Picture:**



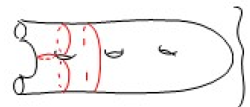

If $\alpha$ is not pants curve and is separating Then

$lk(\alpha) \cong$ join of $\mathcal{C}(X_\alpha)$ and $\mathcal{C}(Y_\alpha)$
$\cong \mathcal{C}(X_\alpha) * \mathcal{C}(Y_\alpha)$

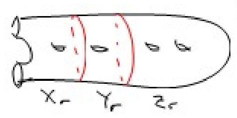
**Note:** If $\alpha$ is pants curve or nonsep
$\mathrm{diam}(lk(\alpha)) \geq 3$ [Assuming $\xi(S) \geq 3$]
If not $\mathrm{diam}(lk(\alpha)) = 2$ [ " $\xi(S) \geq 3$].

**Def:** If $\sigma \in \mathcal{C}(S)$ say that $\sigma$ has $\left\{\begin{matrix} \text{large} \\ \text{small} \end{matrix}\right\}$ link as $\mathrm{diam}(lk(\sigma)) \left\{\begin{matrix} \geq 3 \\ \leq 2 \end{matrix}\right\}$

**Picture.**



$\sigma$ has large link.




$\sigma$ has small link.

**So:** Combinatorial data in $\mathcal{C}(S)$ can determine topological data.

---

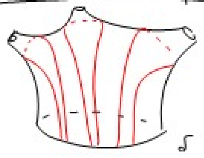
**Recall:** If $i(\alpha, \beta) = 1$ say $\alpha, \beta$ dual.
If $\alpha, \beta$ pants curves and $i(\alpha, \beta) = 2$ say $\alpha, \beta$ dual.

---

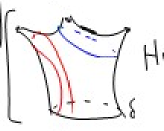
**Def:** Suppose $\partial S \neq \emptyset$, $\delta \subseteq \partial S$ component
$\mathcal{A}(S, \delta) = \{\alpha \mid \partial\alpha \subset \delta, \alpha \text{ an arc}\} / \text{isotopy}$
with simplices given by disjointness.

---

**Ex.**




Exercise
$\mathcal{A}(S_{0,4}, \delta) \cong \mathcal{F}$

[ 

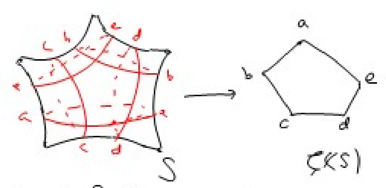
Hint ]

**Exercise.** $\mathcal{A}(S, \delta)$ is connected.
[Surgery Lemma]

---

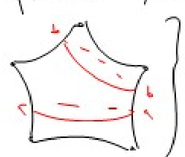
**Lemma** [Pentagon][Korkmaz] $S = S_{0,5}$
The pentagon in $\mathcal{C}(S_{0,5})$ is unique, up to action of $MCG(S)$.

**Pf:** Picture

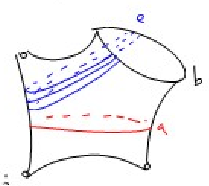



Suppose that $\{a, b, c, d, e\}$ form a pentagon.

Top. Class. of curves gives a homeo. so that

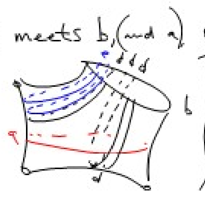



In $T_b$ we see



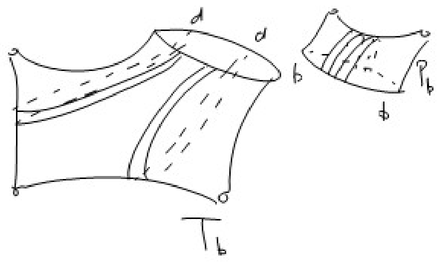

Since $e \cap a = \emptyset$ and $e$ meets $b$ we find:

**Now:** $d$ meets $b$, (and $a$) and is disjoint from $e$.



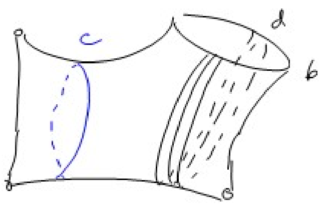

At least one arc of $d \cap T_b$ meets $a$, as shown. We may have arcs of $T_b \cap d$ parallel to arcs of $e \cap T_b$.

If $d\cap T_b$ has arcs parallel to $e\cap T_b$ then $d$ and $b$ fill $S$.

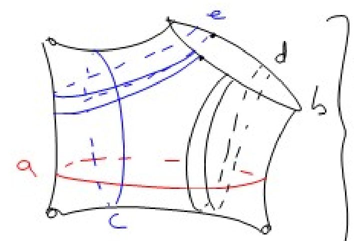


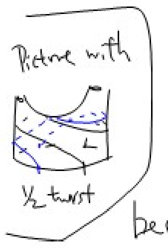

But $i(c,d)=i(c,b)=0$.
So $b,d$ do not fill.
So find $d\cap T_b$ has no arcs parallel to $e\cap T_b$

So see:

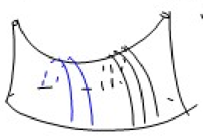



find:

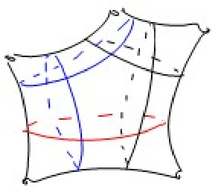



In $P_b$ after ½ twists we see: [No interleaving b/c no interleaving of intersection with $b$]

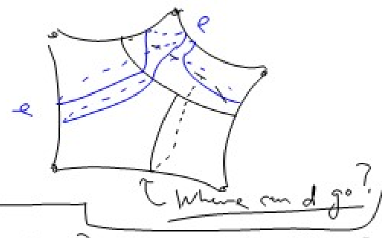


because $d,e$ do not interleave. Since $d,e$ are connected, we find $|d\cap T_b|=|e\cap T_b|=1$ i.e.



As desired. // Pentagon.

Here is a picture that is not allowed by the combinatorics:




---

Corollary: $a,b\in \mathcal{C}(S)$ are dual iff $\exists$ pentagon $P$ in $\mathcal{C}(S)$ s.t. $a,b\in P$ and are not adjacent.

Pf: Exercise.

Corollary: $\mathrm{Aut}(\mathcal{C}(S_{0,5}))$ preserves dual loops.

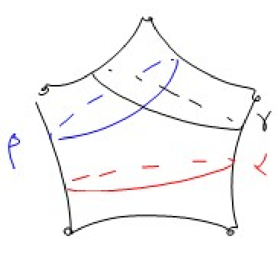
---

Rmk: We must deal carefully with $\mathcal{C}(S_{0,5})$ because it is the base case of Ivanov's Thm in genus $=0$. [Recall: Ivanov $\mathrm{Aut}(\mathcal{C})\cong \mathcal{MCG}(S)$]

The four-holed sphere is not the base case because $\mathcal{C}(S_{0,4})$ was defined via duality not disjointness.

However: The last corollary tells us that we can use the lemma $\mathrm{Aut}(\mathcal{F})\cong PGL(2,\mathbb{Z})$ anyway!

Eg: If $\phi\in \mathrm{Aut}(\mathcal{C}(S_{0,5}))$ fixes $\alpha$ then $\phi|_{\mathrm{lk}(\alpha)}$ is given by some homeo of $T_\alpha$.




No Farey edges in $\mathcal{C}(S_{0,5})$ but $\mathcal{F}(T_\alpha)$ still preserved by Corollary.