Let $S = S_{0,5}$.

<u>Thm</u> [Korkmaz] $\mathrm{Aut}(\mathcal{C}(S)) \cong \mathrm{MCG}(S)$.

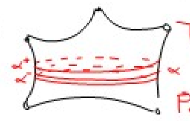
<u>Pf</u>: Fix $f \in \mathrm{Aut}(S)$. Fix $\alpha \in \mathcal{C}(S)$, a basepoint.

<u>Fixing α</u>: By the top. class. of loops in S $\exists f_\alpha \in \mathrm{MCG}(S)$ s.t. $f_\alpha \circ f(\alpha) = \alpha$

<u>Fixing $\mathrm{lk}(\alpha)$</u>: Since $f_\alpha \circ f \in \mathrm{Aut}(\mathcal{C}(S))$

$f_\alpha \circ f$ preserves duality as well as disjointness of loops. So $f_\alpha \circ f \,|\, \mathrm{lk}(\alpha) \in \mathrm{Aut}(\mathcal{F}(T_\alpha))$

<u>Picture</u>:



Let f_{lk} be a homeo of T_α s.t. $f_{lk}(\alpha^+) = \alpha^+$ and

$f_\alpha \circ f \,|\, \mathrm{lk}(\alpha) = f_{lk}^{-1}$. [This is the "induction" step

<u>NB</u>: Need to use hyperelliptics to ensure α^+ fixed]

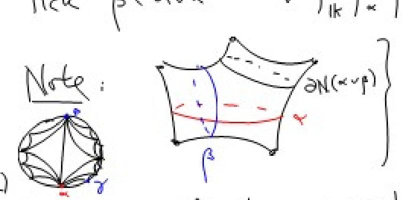
Now extend f_{lk} to a homeo of S, reflecting P_α if necessary. Now $f_{lk} \circ f_\alpha \circ f \,|\, \{\alpha\} \cup \mathrm{lk}(\alpha) = \mathrm{Id}$.

<u>Fixing $\mathrm{dual}(\alpha)$</u>.

<u>Def</u>: $\mathrm{dual}(\alpha) = \{ \beta \mid \alpha, \beta \text{ are dual} \}$.

Pick $\beta \in \mathrm{dual}(\alpha)$. $\gamma = f_{lk} \circ f_\alpha \circ f(\beta)$ is dual to α

<u>Note</u>:



γ is disjoint from ∂N, again because $f_{lk} \circ f_\alpha \circ f \,|\, \mathrm{lk}(\alpha) = \mathrm{Id}$.

$\therefore$ γ and β differ by some number of half twists.

<u>So</u>: Define $\phi = T_\alpha^{k/2} \circ f_{lk} \circ f_\alpha \circ f$.

This has $\phi \,|\, \{\alpha, \beta\} \cup \mathrm{lk}(\alpha) = \mathrm{Id}$.

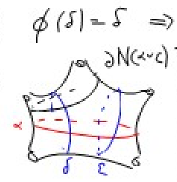
<u>Claim</u>: $\phi = \mathrm{Id} \in \mathrm{Aut}(\mathcal{C}(S))$

The theorem follows from the claim: b/c $f = \underbrace{(T_\alpha^{k/2} \circ f_{lk} \circ f_\alpha)^{-1}}_{\in\, \mathrm{MCG}(S)}$.

<u>Duals</u>: $\phi \,|\, \mathrm{dual}(\alpha) = \mathrm{Id}$. [Crawl in $\mathrm{dual}(\alpha)$]

<u>Pf</u>: <u>Step 1</u>: If δ, ε are dual to α, $\delta \cap \varepsilon = \emptyset$ <u>and</u> $\phi(\delta) = \delta \Rightarrow \phi(\varepsilon) = \varepsilon$.

<u>Pf</u>:

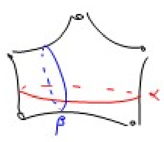


$\phi(\varepsilon) \in \mathrm{dual}(\alpha)$ and $\phi(\varepsilon) \cap \partial N(\alpha \cup \varepsilon)$ is empty. <u>And</u> $\phi(\varepsilon) \cap \delta$ is empty

$\therefore$ $\phi(\varepsilon) = \varepsilon$. //

<u>Step 2</u>: $\mathrm{dual}(\alpha)$ is connected.

<u>Pf</u>: There is a map



$$\begin{array}{ccc} \beta & \in & \mathrm{dual}(\alpha) \\ \downarrow & & \downarrow \pi_\alpha \\ \beta \cap T_\alpha & \in & \mathcal{A}(T_\alpha, \alpha^+) \end{array}$$

<u>Claim</u>: The fibre of π_α containing β is equal to $\mathbb{Z} \cong \text{fibre} = \{ T_\alpha^{n/2}(\beta) \}_{n \in \mathbb{Z}}$.

<u>Check this.</u> [This is "clearly" contained in fibre.]

Since $\mathcal{A}(T_\alpha, \alpha^+)$ is connected ($\mathcal{A}(T_\alpha, \alpha^+) \cong \mathcal{F}(T_\alpha)$)

It would suffice to know that $\mathrm{fibre}(\pi_\alpha)$ is connected.

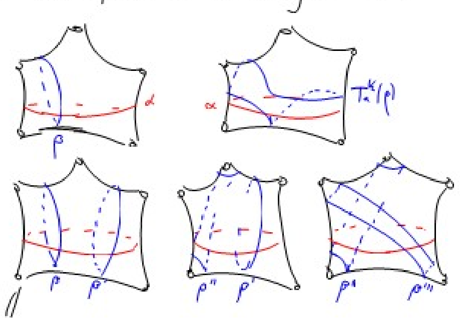
But this is <u>false</u>. $\mathbb{Z} \hookrightarrow \mathrm{dual}(\alpha) \to \mathcal{A}(T_\alpha, \alpha^+)$

[This is why we need both $S_{1,4}$ and $S_{0,5}$ as base cases for the induction]

<u>However</u>: The fibre is <u>relatively connected</u>.

<u>Picture</u>:

[So β fixed $\Rightarrow T_\alpha^{1/2}(\beta)$ also fixed by step 1,]



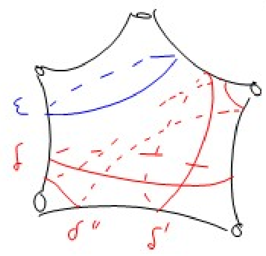
$\therefore$ $\mathrm{dual}(\alpha)$ <u>connected</u> //

So: $\phi \,|\, \{\alpha\} \cup lk(\alpha) \cup dual(\alpha) = Id$.

Crawl in $\mathcal{C}(S_{0,5})$:

If $\delta, \varepsilon \in \mathcal{C}(S_{0,5})$, and $\varepsilon \in lk(\delta)$ and $\phi \,|\, \{\delta\} \cup lk(\delta) \cup dual(\delta) = Id$ then $\phi \,|\, \{\varepsilon\} \cup lk(\varepsilon) \cup dual(\varepsilon)$.

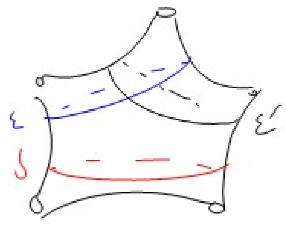
Pf:



Choose a triangle $\delta, \delta', \delta''$ in $\mathcal{F}(T_\varepsilon)$. These are all fixed by ϕ. So

$\phi \,|\, \mathcal{F}(T_\varepsilon) = Id$

$\therefore\ \phi \,|\, lk(\varepsilon) = Id$. Now for the duals.



Pick any $\varepsilon' \in lk(\delta) \cap dual(\varepsilon)$.
$\Rightarrow \phi(\varepsilon') \Rightarrow$ by crawling in $dual(\varepsilon)$ that
$\phi \,|\, dual(\varepsilon)$. //

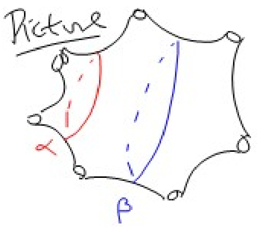
Crawling in $\mathcal{C}(S)$ finishes the proof of Ivanov's Thm for $S_{0,5}$.

NB: This uses the connectivity of $\mathcal{C}(S_{0,5})$ i.e. the surgery lemma.

Thm: $Aut(\mathcal{C}(S_{0,b})) = MCG(S_{0,b})$

Recall: $\alpha \in S$ is a pants curve iff we see

Picture



α, γ dual if both are pants curves and $i(\alpha, \gamma) = 2$.

Exercise: duality is combinatorially determined (Hint: Pentagons)

Steps: Pick $f \in Aut(\mathcal{C}(S))$.

Fix α, Fix $lk(\alpha)$, Pick dual $\beta \in dual(\alpha)$

Fix β, $\phi = T_\alpha^{n/2} \circ f_{lk} \circ f_\alpha \circ f$

Crawl in $Dual(\alpha)$ [Exercise: $Dual(\alpha)$ connected]

Crawl in $\mathcal{C}(S_{0,b})$. //

Exercise: Provide the complete proof.