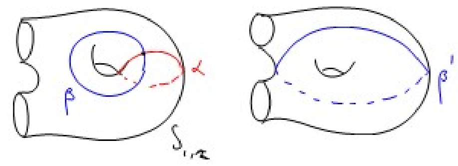
Finishing Luo's Lemma.

Picked $f_\alpha, f_{1k}, T_\alpha^n$ s.t.

$\phi = T_\alpha^n \cdot f_{1k} \cdot f_\alpha \cdot f$ fixes $\{\alpha, \beta\} \cup lk(\alpha)$.

Crawl in dual(α): Step 1, as before, if $\gamma, \delta \in$ dual(α) and $\gamma \cap \delta = \emptyset$ and $\phi(\gamma) = \gamma$ then $\phi(\delta) = \delta$. Pf as before [Pictures]

Step 2: dual(α) is not connected.



As usual there is a fibration

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathrm{dual}(\alpha) & \longrightarrow & \mathcal{A}(S_\alpha; \alpha^\pm) \\ \cong & & & & \| \\ \{T_\alpha^n(\beta)\}_{n \in \mathbb{Z}} & & & & \{\text{arcs in } S_\alpha \text{ joining } \alpha^\pm\} \end{array}$$

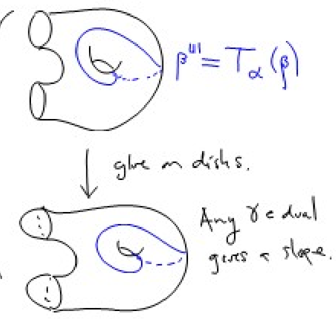
Exercise: $\mathcal{A}(S_\alpha, \alpha^\pm)$ is connected.

In $S_{1,2}$ the fibre $\{T_\alpha^n(\beta)\}$ is not rel. conn.

Picture: Consider the "capping off" map

dual(α) $\longrightarrow \hat{\mathbb{Q}}$

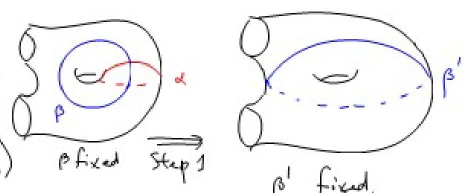
If dual(α) conn. then $\exists$ path of disjoint slopes connecting β to β''' in T^2 $\divideontimes$

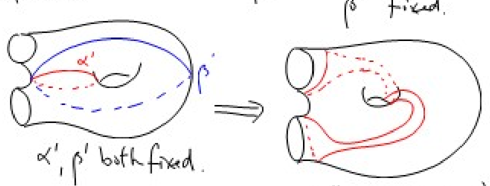


Lemma: If $\gamma = T_\alpha^n(\beta)$ and $\delta = T_\alpha^{n+1}(\beta)$ and $\phi(\gamma) = \gamma$ then $\phi(\delta) = \delta$.

Pictures:

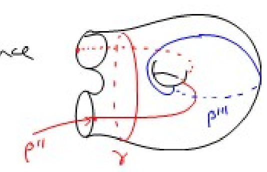
{ Rmk: suffices to prove that $\beta''' = T_\alpha(\beta)$ fixed }





$\beta'' = \partial N(\alpha' \cup \beta')$ fixed.

Hence



As β''' is disjoint from β'' and disjoint from γ, β''' is also fixed. // Crawling

Now crawl in $\mathcal{C}(S_{1,2})$

Claim: If ϕ fixes $\{\gamma\} \cup lk(\gamma) \cup$ dual(γ) and $\delta \in$ dual(γ) $\Rightarrow \phi$ " $\{\delta\} \cup lk(\delta) \cup$ dual(δ).

Pf: Exercise. //

Now we are done // Luo Lemma.

Finally: Ivanov's thm for $g \geq 1$ has similar (easier) pf to Luo's Lemma. //

Exercise: Prove

Thm: If $\mathcal{C}(S) \cong \mathcal{C}(\Sigma)$ (isomorphic) then

(i) $S \cong \Sigma$ (homeomorphic) or

(ii) $\{S, \Sigma\} = \{S_2, S_{0,6}\}$

(iii) $\quad\quad = \{S_{1,2}, S_{0,5}\}$

(iv) $\quad\quad \subseteq \{S_{1,1}, S_1, S_{0,4}\}$

(v) $\quad\quad \subseteq \{\mathbb{S}, \mathbb{D}, \mathbb{A}, S_{0,3}\}$.

Hint: classification of surfaces

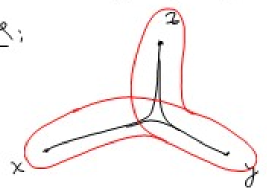
Now discuss two papers by Masur Minsky.

I) Thm: $\forall S, \exists \delta = \delta(S)$ so that $\mathcal{C}(S)$ is δ-hyperbolic. [Masur Minsky Another pf by Bowditch]

Def: A geodesic metric space (X, d_X) is δ-hyp. if $\forall x, y, z \in X$ any geodesic triangle $[x,y] \cup [y,z] \cup [z,x]$ is δ-slim:

That is $N_\delta([x,y] \cup [y,z]) \supseteq [z,x]$

picture:



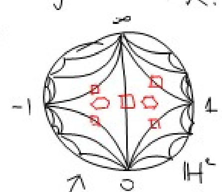
Ex: ① $\mathbb{E}^2$ is not δ-hyp for any δ.
② Trees are 0-hyp. [0-hyp iff tree]
③ $\mathbb{H}^n$ (equiv $\mathbb{H}^2$) is $\log\left(\frac{1+\sqrt{2}}{2}\right)$-hyp.

Tools for proving Thm (I):

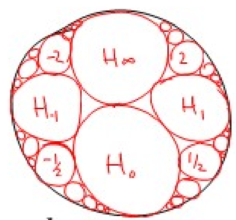
(i) $\mathrm{Teich}(S) = \mathcal{T}(S)$ = Teich. space

(ii) $\mathrm{sys}: \mathcal{T}(S) \to \mathcal{C}(S)$
$(f, X) \longmapsto$ shortest simple closed geodesic in X.

Example: $\mathcal{T}(\mathbb{T}^2) \cong \mathbb{H}^2$

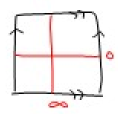


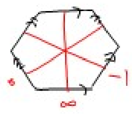
Drawn the square and hexagonal tori in $\mathcal{T}(\mathbb{T}^2)$



Apollonius Packing

In the $\left\{\begin{matrix}\text{square} \\ \text{hexagon}\end{matrix}\right\}$ $\exists$ $\left\{\begin{matrix}\text{two} \\ \text{three}\end{matrix}\right\}$ shortest curves





Thus: The systole map is not well-defined but is coarsely well-defined. (The set of systoles always has bounded diameter in $\mathcal{C}(S)$.

In general: $H_\alpha^\varepsilon = \{(f, X) \mid \ell_X(f(\alpha)) \leq \varepsilon\}$

If ε is small enough then the nerve of $\{H_\alpha^\varepsilon\}_{\alpha \in \mathcal{C}(S)}$ is $\mathcal{C}(S)$. [collar lemma]

Tool (iii): Teichmuller geodesics track $\mathcal{C}(S)$-geodesics. //