THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: JUNE 2008

KNOT THEORY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

1. a) Define the Reidemeister moves on a link diagram. **[6]**

b) Define the *sign* of a crossing of an oriented diagram. **[2]**

c) Define the *linking number* of components C and C' of an oriented diagram. **[5]**

d) Prove that the linking numbers of the components of a link are isotopy invariants. **[7]**

e) Give diagrams for the $(2,4)$–torus link and the Whitehead link. Prove that these links are not isotopic. **[5]**

2. a) Define the *winding number* of an oriented diagram around a region of the diagram. **[5]**

b) Prove that the winding number is well-defined. **[7]**

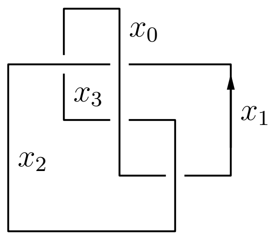

Figure 1:

c) Compute the winding numbers for the diagram shown in Figure 1. **[3]**

d) Find a matrix of crossing equations for the labelled knot given in Figure 1. Use the winding numbers computed in part (c) to ensure that each row sums to zero and each column sums to zero. **[7]**

e) Compute the Alexander polynomial of the knot shown in Figure 1, using part (d). **[3]**

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3. a) Give generators and relations for B_n: the braid group on n strands. Discuss the connection between the relations in the braid group and the Reidemeister moves. **[7]**

b) Describe what it means for a knot to be obtained via *braid closure*. **[5]**

c) Define the *Seifert circles* of an oriented diagram. **[3]**

d) Describe an algorithm for converting an oriented link into the closure of a braid. You need not prove that the algorithm works. **[5]**

e) Convert the diagram of Figure 2 to a closed braid using the algorithm above. Record the resulting braid word. Show the intermediate Seifert circles. **[5]**

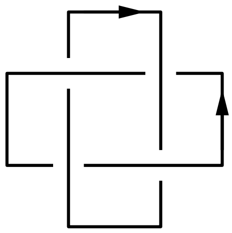

Figure 2:

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4. a) Define the *bracket polynomial* of an unoriented diagram. **[7]**

b) Define the *Kauffman polynomial* for oriented links. Define the *Jones polynomial* for oriented links. **[7]**

c) Compute the Jones polynomial for L_n, the link of n components shown in Figure 3. Clearly state any theorems you use from class. **[11]**

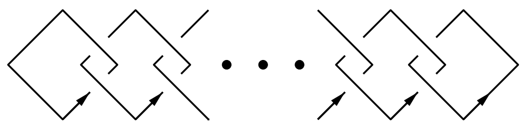

Figure 3:

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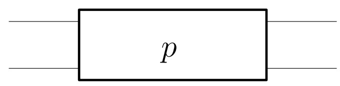
5. a) Define a *twist box*, containing p twists. **[3]**



b) Define a *flype*. **[3]**

c) Describe an isotopy between links having the diagrams shown in Figure 4. **[8]**

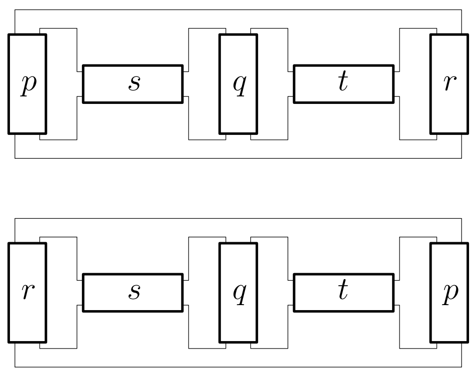


Figure 4:

d) In case $r = p = 1$ identify the knot with the numerator $N(T)$ of a tangle T. Identify the rational number associated with T. **[11]**

MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS

Course Title: KNOT THEORY

Model Solution No: 1

a) Define the Reidemeister moves on a link diagram. **[6]**

There are four Reidemeister moves: R_0 is planar isotopy. Each of R_0, R_1, and R_3 is best shown via a picture: R_0 takes an arc and adds a twist. R_1 takes two arcs and passes a segment of one under the other. R_3 takes three arcs and passes a segment of the first under a cross of the other two. (R_0 is not required for full marks.)

b) Define the *sign* of a crossing of an oriented diagram. **[2]**

Fix a crossing c of the diagram. Rotate the xy–plane so that the arcs point northeast and northwest. When the overcrossing arc points northeast (has positive slope) then $\text{sign}(c) = +1$. When the overcrossing arc points northwest (has negative slope) then $\text{sign}(c) = -1$.

c) Define the *linking number* of two components C and C' of an oriented diagram. **[5]**

The *linking number* of C and C' is

$$\text{lk}(C, C') = \frac{1}{2} \sum_c \text{sign}(c)$$

where the sum is taken over the crossings between C and C'.

d) Prove that the linking numbers of the components of a link are isotopy invariants. **[7]**

By a theorem proven in class it suffices to check that the Reidemeister moves leave $L = \text{lk}(C, C')$ unchanged. One picture for each move suffices to check the claim. In fact only the R_2 move may change the number of terms in the sum defining $\text{lk}(C, C')$.

R_1 adds of adds or subtracts exactly one crossing of some component with itself. R_2: if the two arcs involved are not exactly one from C and one from C' then no crossings between C and C' are affected. If an arc from C is passed under an arc from C' then two terms, with opposite signs, are added to the sum. The other case is similar. R_3: there is a bijection between crossings before and after the move which preserves sign.

e) Give diagrams for the $(2, 4)$–torus link and the Whitehead link. Prove that these links are not isotopic. **[5]**

The former has linking number ± 2, depending on orientation. The Whitehead has linking number zero.

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Course Title: KNOT THEORY

Model Solution No: 2

a) Define the *winding number* of an oriented diagram around a region of the diagram. **[5]**

Suppose that R is a region of the diagram D. Pick any point $x \in R$ and choose any ray L emanating from x which is transverse to D and disjoint from the crossings. Define $w(R)$, the winding number of D about R, to be the sum over intersections $L \cap D$ where a crossing arc contributes $+1$ if it crosses from right to left and -1 if it crosses from left to right.

b) Prove that the winding number is well-defined. **[7]**

If L' is another ray, then we may connect L to L' by a continuous family $\{L_t\}$ of rays. The defining sum only changes when L_t is not transverse or goes through a crossing. Passing through a crossing rearranges the terms of the sum. Passing through a tangency either adds or removes a pair of terms of opposite sign from the sum. (Strictly speaking there are other tangencies possible, such as inflection points. But these need not be mentioned to receive full marks.)

If y is another point in R then connect x to y by a polygonal path, contained in R, with vertices $x = x_0, x_1, \ldots, x_n = y$ and with $n \geq 2$. Choose the x_i $(i \neq 0, n)$ so that the line through $[x_i, x_{i+1}]$ is transverse to D and avoids the crossings. Then the ray based at x_i, through x_{i+1}, defines the same winding number as any other ray based at x_{i+1}. Thus, the winding numbers at x_i and x_{i+1} agree.

c) Compute the winding numbers for the diagram shown in Figure 1. **[3]**

This is done it Figure 5.

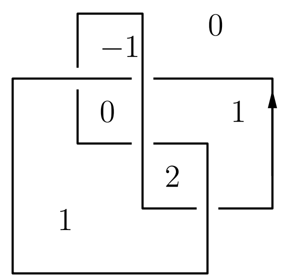

Figure 5:

d) Find a matrix of crossing equations for the labelled knot given in Figure 1. Use the winding numbers computed in part (c) to ensure that each row sums to zero and each column sums to zero. **[7]**

Label each region R with the monomial $t^{w(R)}$. Around every crossing of the diagram draw a small square and orient the edges from the region with lower winding number towards the region with higher. For every edge of the square record $\pm t^w$ in the matrix. Here the sign is is $+1$ if the orientation on the edge agrees with the counterclockwise orientation of the square. Also, the power w is the winding number of the region pointed at by the edge. Finally, the monomial is placed in the row of the crossing and in the column of the overcrossing arc.

Note that a row may be multiplied or divided by $\pm t^n$ and still yield the correct Alexander polynomial, up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$. However, to get row and column sums to be zero, the above or some similar convention must be followed.

For the given knot we find:

$$A_+ = \begin{bmatrix} t-1 & -t & 1 & 0 \\ t^2 - t & 0 & -t^2 & t \\ -t^2 & t & t^2 - t & 0 \\ 1 & 0 & t-1 & -t \end{bmatrix}.$$

e) Compute the Alexander polynomial of the knot shown in Figure 1, using part (d). **[3]**

We must delete any one row and any one column of the matrix A_+ and then take the determinant. For the trefoil this yields $\Delta_K(t) = t - 1 + 1/t$.

MATHEMATICS DEPARTMENT
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Course Title: KNOT THEORY

Model Solution No: 3

a) Give generators and relations for B_n: the braid group on n strands. Discuss the connection between the relations in the braid group and the Reidemeister moves. **[7]**

B_n is generated by the elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with relations

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i - j| \geq 2$ and
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $i \leq n - 2$.

We see that the first (far commutivity) follows from the R_0 move (planar isotopy). The second is the same as R_3. Also, $\sigma\sigma^{-1} = 1$ follows from the R_2 move.

b) Describe what it means for a knot to be obtained via *braid closure*. **[5]**

The strands of the braid σ, drawn in the plane, connect the set of points $\{(i, 0)\}$ to the points $\{(i, 1)\}$ for $i = 1, 2, \ldots, n$. In the complement of the diagram, without adding additional crossings, for each i add a strand connecting $(i, 1)$ to $(i, 0)$. The resulting link L_σ, is the braid closure of σ.

c) Define the *Seifert circles* of an oriented diagram. **[3]**

Suppose that the diagram D is given. *Smooth* every crossing as shown in Figure 6. The new diagram has no crossings and gives a collection of oriented circles in the plane.

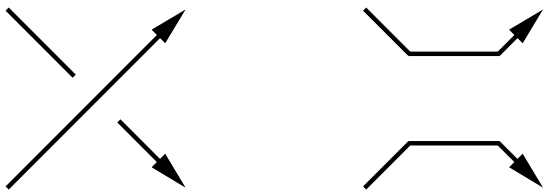

Figure 6:

d) Describe an algorithm for converting an oriented link into the closure of a braid. You need not prove that the algorithm works. **[5]**

Suppose that D' is the smoothing of D. A *conflict* for a diagram is an arc α so that α meets D only in its endpoints, α meets distinct Seifert circles $C, C' \subset D'$, and the orientations of C, C' disagree. (Thought of as lying in S^2.) We *resolve* the conflict by performing an R^2 move along α. This gives a new diagram.

Here is the algorithm: As long as there are conflicts, resolve them. (Optional, additional steps: Perform R_∞ moves to ensure that the Seifert circles are nested. Use an R_0 move to make the diagram look like a braid closure.)

e) Convert the diagram of Figure 2 to a closed braid and record the resulting braid word. Show the intermediate Seifert circles. **[5]**

The first three lines of Figure 7 show the initial link and its circles, and the result of performing the first and then second R_2 move.

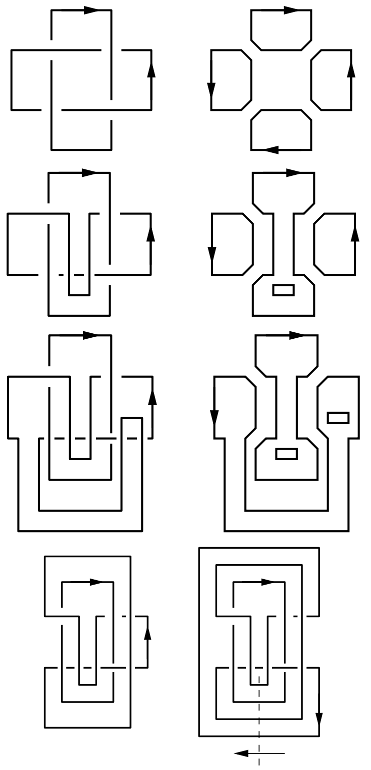

Figure 7:

The last line shows the effect of a pair of R_∞ moves. We arbitrarily insert a beginning to the braid at the dotted line. The strands are numbered 1, 2, 3, and 4 from the outside in. The braid word can now be read off:

$$\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-1}\sigma_1.$$

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Model Solution No: 4

a) Define the *bracket polynomial* of an unoriented diagram. **[7]**

Suppose that c is a crossing of diagram D. Let D_R and D_L be the diagrams obtained by performing a right and left smoothing at c, respectively. Then the bracket polynomial is inductively defined by

$$\langle D\rangle = A\langle D_R\rangle + B\langle D_L\rangle$$

with base conditions

$$\langle O\rangle = 1, \quad \langle O \sqcup D\rangle = C\langle D\rangle.$$

To ensure that $\langle\cdot\rangle$ is a regular isotopy invariant we set $B = A^{-1}$ and $C = -(A^2 + A^{-2}$.

b) Define the *Kauffman polynomial* for oriented links. Define the *Jones polynomial* for oriented links. **[7]**

Recall that the writhe of D is $w(D)$: the sum of the signs of the crossings of D. The Kauffman polynomial for K is defined by

$$X_K(A) = (-A^{-3})^{w(D)}\langle D\rangle$$

where D is any diagram of K.

The Jones polynomial is defined by $V_K(t) = X_K(t^{-1/4})$.

c) Compute the Jones polynomial for L_n, the link of n components shown in Figure 3. Clearly state any theorems you use from class. **[11]**

The chain is the connect sum of $n-1$ copies of the right Hopf link. By a theorem from class, the Jones polynomial is multiplicative on connect sums. Thus $V_{L_n}(t) = (V_H(t))^{n-1}$, where H is the right Hopf link. To compute the Jones polynomial of H we use the skein relation

$$t^{-1}V_+ - tV_- = (t^{1/2} - t^{-1/2})V_0$$

and the fact that

$$V_{O\sqcup D} = -(t^{1/2} + t^{-1/2})V_D.$$

We find that

$$\begin{aligned} t^{-1}V_H - tV_{OO} &= (t^{1/2} - t^{-1/2})V_0 \\ t^{-1}V_H - t(-(t^{1/2} + t^{-1/2})) &= t^{1/2} - t^{-1/2} \\ V_H + t^2(t^{1/2} + t^{-1/2}) &= t(t^{1/2} - t^{-1/2}) \\ V_H &= -t^{1/2} - t^{5/2}. \end{aligned}$$

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Course Title: KNOT THEORY

Model Solution No: 5

a) Define a *twist box*, containing p twists. **[3]**

The box is a sequence of $|p|$ right-handed twists if p is positive (left-handed if p is negative), connected end-to-end.

b) Define a *flype*. **[3]**

A flype commutes a tangle past a crossing, at the cost of rotating the tangle by 180 degrees. The rotation is around the line connecting the center of the crossing with the center of the tangle.

c) Describe an isotopy between links having the diagrams show in Figure 4. **[8]**

Rotate the first figure by 180 degrees about its center. Now the twist boxes, in order, have sizes r, t, q, s, p. Commute $s - t$ twists past the q–box using flypes.

d) In case $r = p = 1$ identify the link with the numerator $N(T)$ of a tangle T. Identify the rational number associated with T. **[11]**

Note that a vertical twist box of size 1 is identical to a horizontal twist box of size -1. So, combining the two (-1)–boxes so formed with the s and t–boxes, we find a link with boxes of size $s - 1$, q, and $t - 1$. Now flype the $s - 1$ twists past the q–box to find a link with boxes of size q and $s + t - 2$. This is the numerator $N(q, 2 - s - t)$ and corresponds to the rational number $2 - s - t + \frac{1}{q}$.