Knot Theory MA3F2

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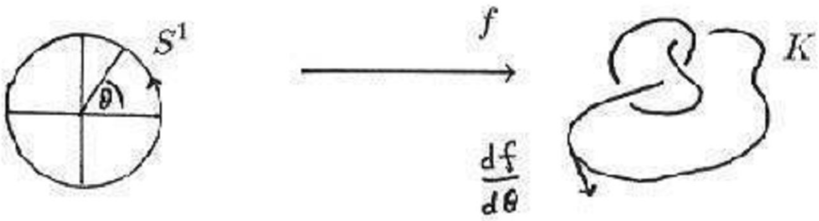
1 Knots links and diagrams

Definition

A subspace $K \subset \mathbb{R}^3$ is a knot if it is the image of a smooth injective map $f\colon S^1 \to \mathbb{R}^3$, with $\frac{df}{d\theta}$ never zero.



Smooth means $\frac{d^n f}{d\theta^n}$ exists for all n.

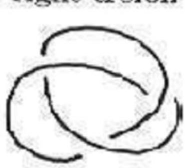
Example 1.1 Not a knot since $\frac{df}{d\theta}$ is undefined at * .

Definition A subspace $L \subset \mathbb{R}^3$ is a link if $L = K_1 \cup \ldots \cup K_n$ for some knots K_i and $K_i \cap K_j = \emptyset$ if $i \neq j$. Say L has n *components* and write $n = comp(L)$. Notice in particular that a knot is a link with one component.

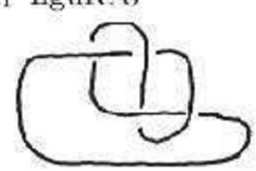
We study links via their *diagrams*: pictures of projections in $\mathbb{R}^2 = \{(x_1, x_2, 0) \in \mathbb{R}^3 | x_i \in \mathbb{R}\}$.

unknot:  unlink with $comp(L) = 4$: 

3_1 right trefoil



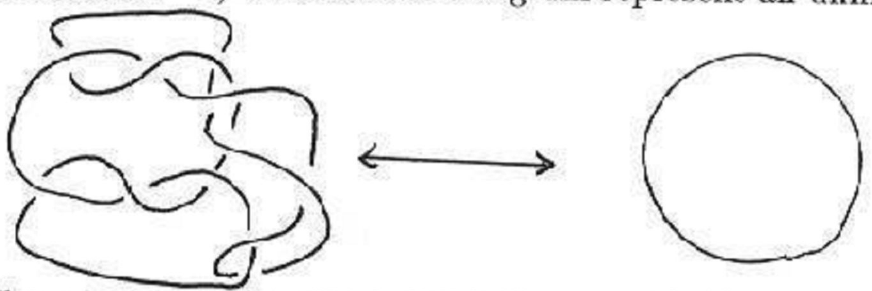
4_1 figure 8



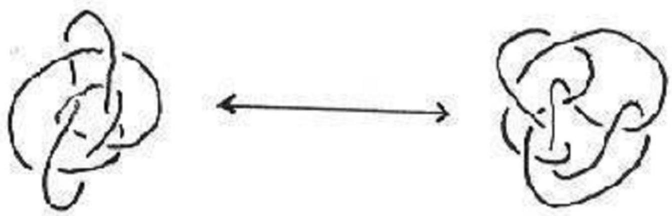
Hopf Link

Whitehead link

Problems 1.2 i) When does a diagram represent an unlink?

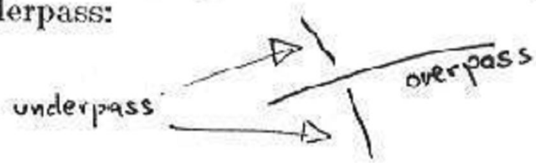


ii) When do two diagrams represent the same link?



There are 12,965 knots with ≤ 13 crossings – excluding mirrors and composites.

Definition A *diagram* in $\mathbb{R}^2$ is made up of a number of *arcs* and *crossings*. At a crossing one arc is the overpass and the other two make up an underpass:



Not allowed in a diagram:

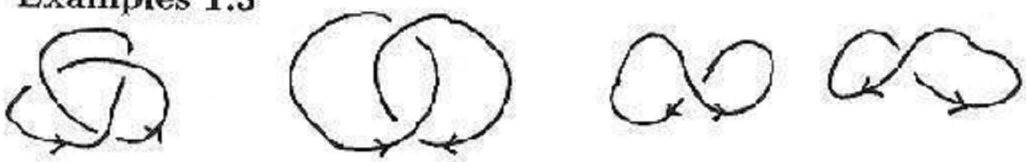
Definition An *oriented* link is determined by choosing a direction along each component indicated on the corresponding diagram by arrow heads. Then we have two types of crossing:

positive/right hand negative/left hand

Definition The *writhe* of an oriented diagram is

$$\omega(D) = \sum_{crossings} \text{sign of crossing} \in \mathbb{Z}$$

Examples 1.3



$\omega(D) = -3$ $\omega(D) = 2$ $\omega(D) = -1$ $\omega(D) = 1$

Remark 1.4 Changing all orientations does not change writhe – so the writhe of an *unoriented* **knot** diagram is well defined.

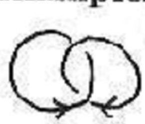
Definition Suppose an oriented diagram D has components $C_1, \ldots, C_n$. Define the *linking number* of C_i with C_j, $i \neq j$, to be

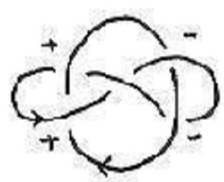
$$lk(C_i, C_j) = \frac{1}{2} \sum_{crossings\ of\ C_i\ with\ C_j} \text{sign of crossing}$$

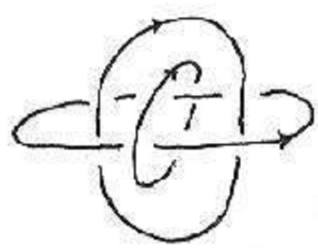
The *linking number* of D is

$$lk(D) = \sum_{1 \leq i < j \leq n} lk(C_i, C_j)$$

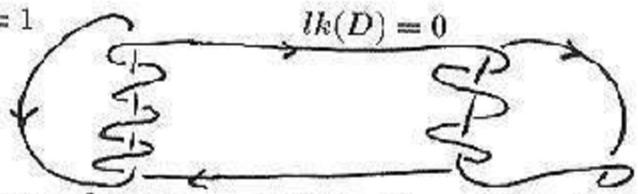
Examples 1.5







$lk(D) = 1$ $lk(D) = 0$ $lk(C_i, C_j) = 0, i \neq j$



$lk(C_1, C_2) = \frac{1}{2}8 = 4$ $lk(D) = 7$ $lk(C_2, C_3) = \frac{1}{2}6 = 3$ $\omega(D) = 15$

Can you see the isotopy between the Perko pair? My favourite pair. Find the writhe and linking number of this celtic earring. The reef knot decomposed into two trefoils - join ends and find the isotopy. Pictures of knots with 10 crossings taken from the tables of Tate and Little. The Perko pair is 5II and 6VI. Here 6VI is omitted.

Here is an unknotting of a `Gordian knot'.

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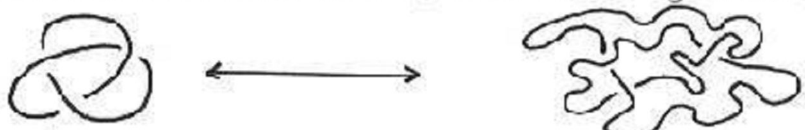
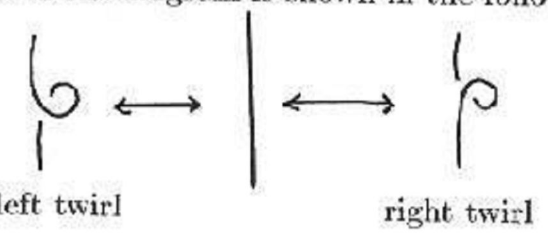
1.6 Diagram moves

R_0 Homotopy of diagrams preserving arc and crossing structure:

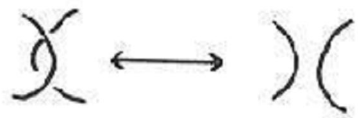


Only the affected part of the diagram is shown in the following moves.

R_1



R_2



R_3

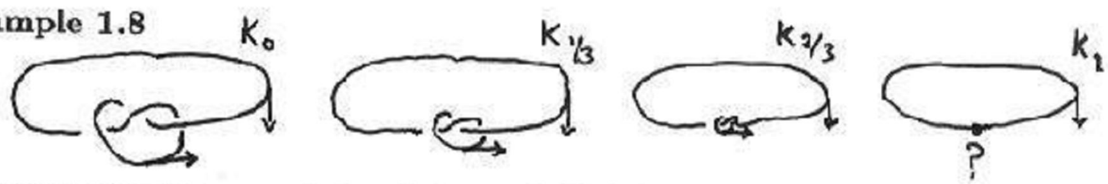
Remark 1.7 The moves R_1, R_2, R_3, are known as *Reidemeister moves*. In view of R_0 we can think of R_3 as moving any one of the strands across the crossing of the other two. The moves are presumed to preserve orientation if orientations are present. Often we use R_0 without explicit mention.

Definition Diagrams D and D' are *isotopic* if D can be obtained from D' by a sequence of moves of type R_i, $0 \leq i \leq 3$. The diagrams are *regularly isotopic* if R_1 is not used.

Definition Knots K_0 and K_1 are isotopic if there exist knots $K_t, t \in [0,1]$, given by a homotopy $f_t \colon S^1 \to \mathbb{R}^3$ so that $f_t S^1 = K_t$, for all t, and $f \colon S^1 \times I \to \mathbb{R}^3$, given by $f(x,t) = f_t(x)$, is smooth.

Definition Links L_0 and L_1 are *isotopic* if ....

Example 1.8



This shows knots cannot simply be pulled tight by an isotopy. To see this consider $\frac{\partial f}{\partial \theta}(\theta, t) = \frac{df_t}{d\theta}$.

1.9 Theorem *(Reidemeister (1932))*

i) Any link L is isotopic to a link which has a diagram.

ie projection to the $x_0\, x_1$ plane gives a diagram

ii) Suppose given links L_0, L_1 with diagrams D_0, D_1. Then L_0 and L_1 are isotopic if and only if D_0 and D_1 and are isotopic.

1.10 Remark The theorem holds with or without orientations.

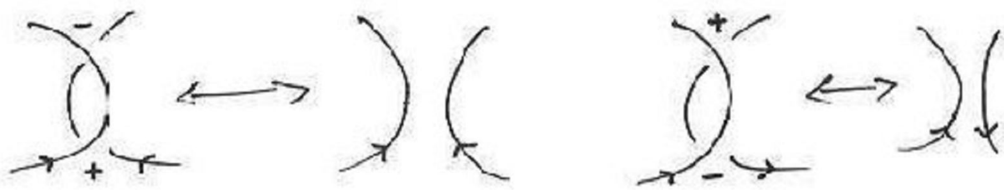
Define the *linking number* of an oriented link to be the linking number of a corresponding oriented diagram.

1.11 Corollary *The linking number is a well defined isotopy invariant.*

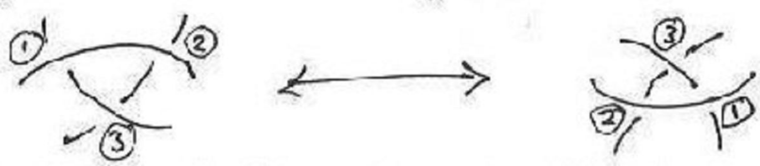
Proof. Check that the R moves do not change linking number:

R_0 and R_1 are not involved in the calculation of linking numbers.

R_2



R_3 The contribution from any two strands remains the same:



1.12 Corollary *Hopf is not isotopic to Whitehead.* □

2 Knot Colouring

Definition A link L can be coloured *mod* n if for some diagram D for L integers can be assigned to arcs as shown:

$a + c = 2b \mod n$

ie overpass arc = 'average' of underpass arcs. Further we do not allow a constant colouring—all arcs the same colour *mod* n . Equivalently we

Some knots from the Alambra

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can work in $\mathbb{Z}_n$ the cyclic group. Note that $\mathbb{Z}_0 \cong \mathbb{Z}$ and $\mathbb{Z}_1 \cong \{0\}$ so 1 is never a colouring number. **Beware** 'average'. **Division by 2 is not always well defined - eg mod 4**

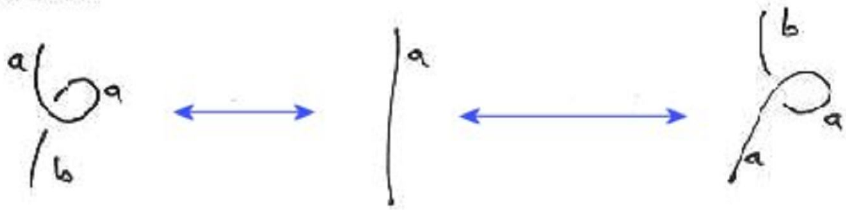
2.1 Examples

2.2 Theorem *If a link L can be coloured mod n, then every diagram for L can be coloured mod n.*

Proof We check that the R moves do not change colourability.

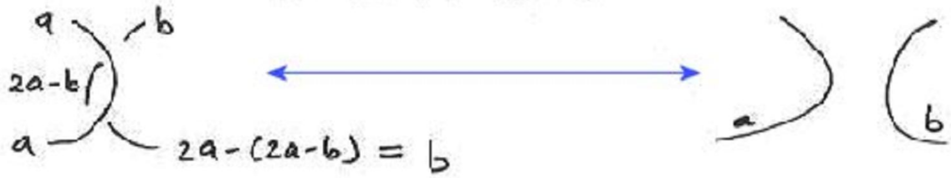
R_0 no problem

R_1



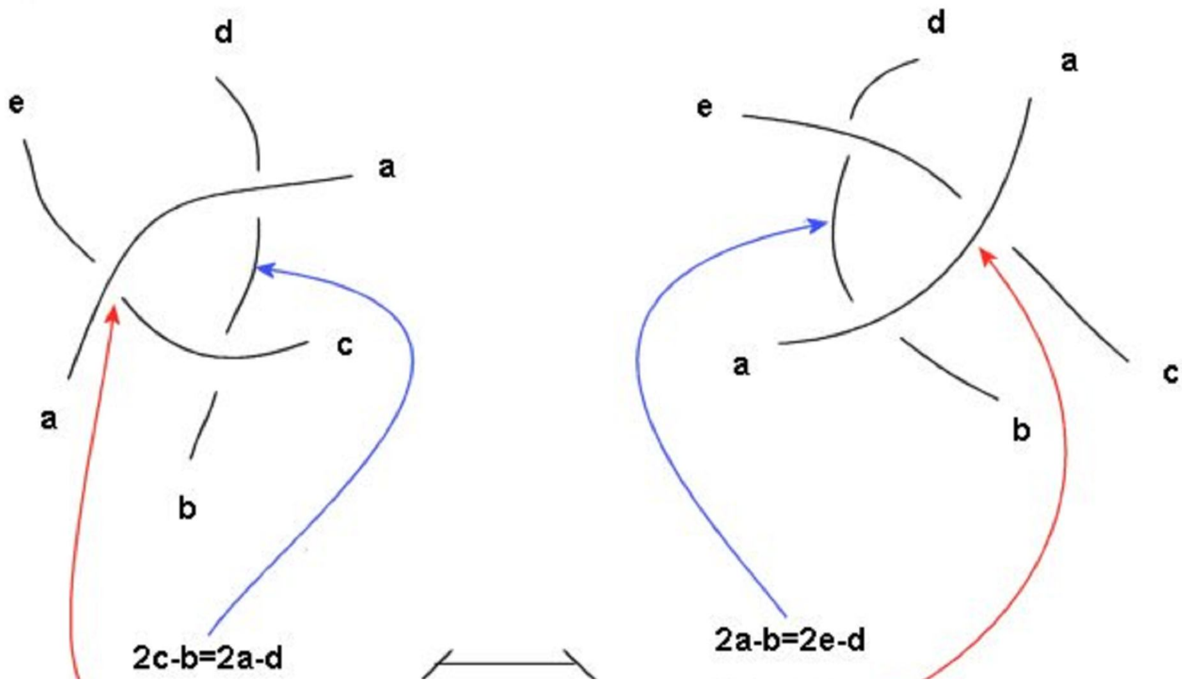
$$2a = a + b \iff b = a$$

R_2



If $a = b$ on the right then $2a - b = a$ so $2a - b$ is not a new colour on the left.

R_3



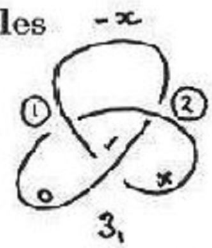
We only checked 1 of the 2 R3 moves, but only one is needed—see examples sheet, number 6.

2.3 Proposition *If a diagram is colourable mod n then it is colourable mod n with prescribed integer (eg 0) on any given arc.*

Proof If $2b = a + c \ \ mod \ \ n$ then $2(k + b) = (k + a) + (k + c) \ \ mod \ \ n$. So add suitable k (to every value) to get the prescribe number on the prescribed arc. □

2.4 Examples

i)



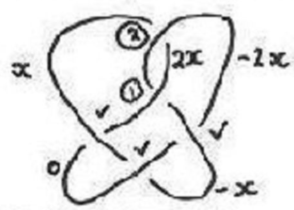
$$① \quad 2(-x) = x + 0$$
$$② \quad 2(x) = -x + 0$$
$$3x = 0$$

We want $x \neq 0 \ \ mod \ \ n$ since $x = 0 \ \ mod \ \ n$ gives the constant colouring. Then $3x = kn$ implies 3 divides n, otherwise 3|k and x is a multiple of n.

So 3_1 is colourable $mod \ \ n$ iff $3|n$. Further 3_1 is knotted (not isotopic to the unknot) since the unknot cannot be coloured for any n.

ii)

5_2

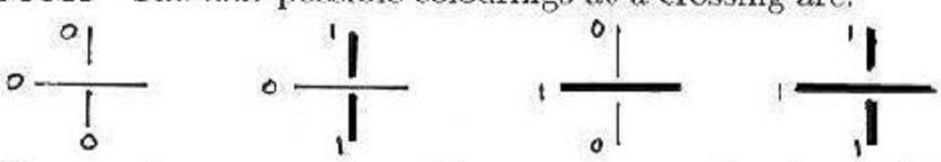


$$① \quad 2(2x) = (-2x) + (-x)$$
$$② \quad 2(-2x) = x + 2x$$
$$7x = 0$$

So 5_2 is colourable $mod \ \ n$ iff $7|n$. Further 5_2 is knotted and not isotopic to 3_1.

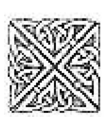
2.5 Proposition *Any link with more than 1 component is colourable mod 2. No knot can be coloured mod 2.*

Proof The four possible colourings at a crossing are:



Hence each component must have a constant colouring and the knot result follows. For a link with more than one component, colour one component with 0 and the others with 1. □

Take a look at the Dunfallandy stone:



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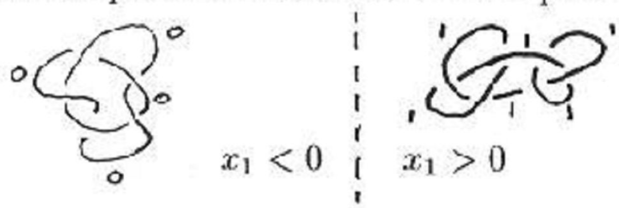
Definition A link or diagram is *splittable* if, possibly after isotopy, it consists of two pieces, one in the region $x_1 < 0$ and the other in the region $x_1 > 0$.

Clearly a link is splittable iff it has a splittable diagram.

2.6 Proposition *If a link is splittable then it can be coloured mod n for any $n \neq 1$.*

Proof Colour one piece with 0 and the other piece with 1.



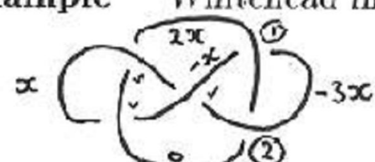
□

2.7 Corollary *The Borromean rings are not splittable.*

Proof Simply check not colourable *mod* 3. □

Here is a colouring of the Borromean rings *mod* 4 using all four colours:

2.8 Example *Whitehead link*



① $2(2x) = (-x) + (-3x)$

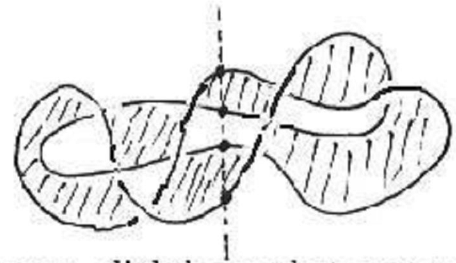
② $2(-3x) = 0 + 2x$

$8x = 0$

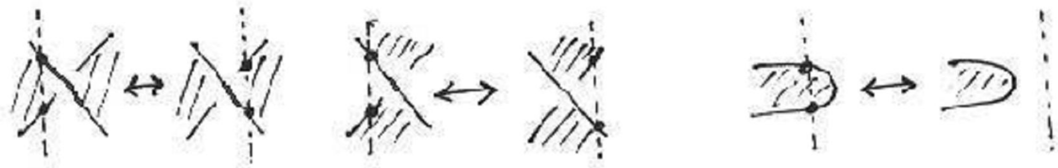
Any non zero even number is a colouring number since $8k = 0$ *mod* $2k$ and $k \neq 0$ mod $2k$. It is easy to see that no odd number is a colouring number, so the set of colouring numbers for this link is $2\mathbb{N} \setminus \{0\}$.

The examples suggest that we get one more equation than we need. This is the case as we shall see next.

2.9 Lemma *Any diagram can be 'chess boarded':*

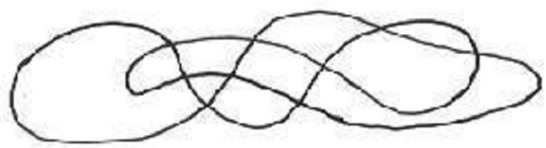


Proof Move the diagram slightly so that any vertical line meets it in only a finite number of points. Given a vertical line mark the regions it passes through alternately 'white' and 'black'. Observe consistency of the white/black choices as the line moves through a crossing or bend.

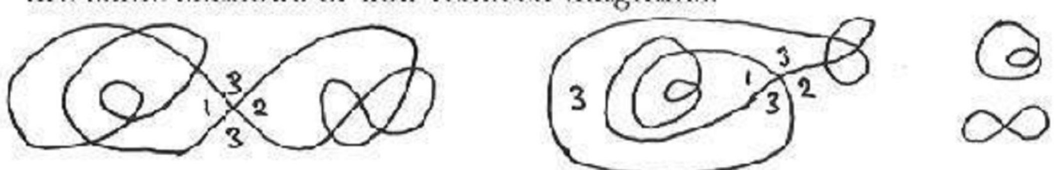


□

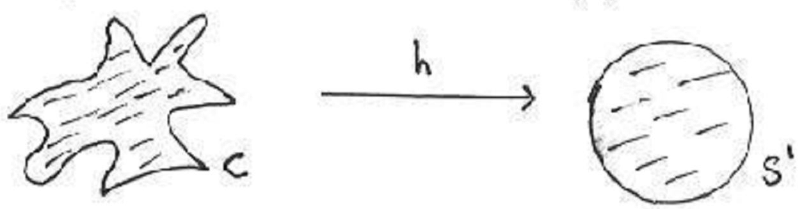
Definitions By ignoring the crossing information in a diagram we get a *shadow* S:



The connected components of the complement of S are the *regions*. A diagram is *reduced* if at each crossing there are four distinct regions. Here are some shadows of non reduced diagrams:



2.10 Theorem *(Schönflies) For any closed curve C in $\mathbb{R}^2$ there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(C) = S^1$.*



□

2.11 Lemma *Regions are path connected.* □

2.12 Proposition *Suppose L is a link. Then (after isotopy) L has a reduced diagram.*

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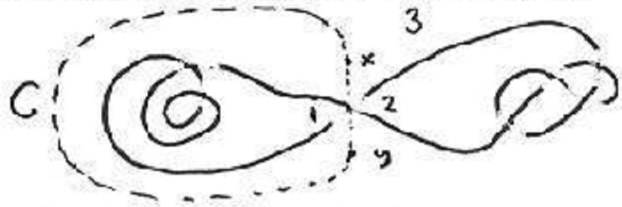
Proof Choose a diagram for L. If it is not reduced we can use the lemma to find a closed curve C as indicated:



We may assume the shadow coincides with L except near crossings.

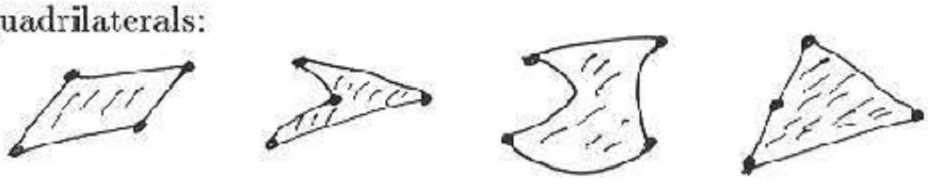
Join x and y by an arc in the bad region then extend across the bad crossing to form C. By the Schöenflies theorem we have a disc inside C. Turn the disc over to eliminate the bad crossing. Repeat until all bad crossings have been eliminated. □

We say a diagram is *connected* if its shadow is connected. Suppose a diagram D is reduced, connected and has no closed curves:

not allowed

Given the proposition it is easy to see that every link has such a diagram. The diagram determines a decomposition of the plane into quadrilaterals—with one exception which is (topologically) a punctured quadrilateral.

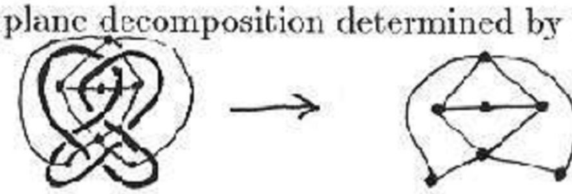
Some quadrilaterals:



A punctured quadrilateral:

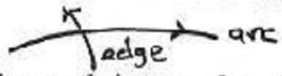
Put in ∞ to repair the puncture.

The plane decomposition determined by a diagram for 5_2:



Place a vertex in each region and 'edges' across arcs. If arcs are divided into short arcs at overcrossings we have one edge for every short arc. Each crossing is now contained in a quadrilateral—except for one which is a punctured quadrilateral. We would not need the exception if diagrams were on the 2-sphere— recall stereographic projection:

Choose an orientation for the diagram. This determines an orientation on each edge by the left hand rule:



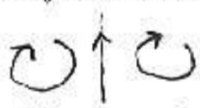
Notice that opposite sides of a quadrilateral have the same direction. Choose a chess boarding then label an edge positive if its head is white, otherwise label the edge negative. For each crossing/quadrilateral write the colouring equation as follows:

$$\pm a \mp b \pm c \mp b = 0$$

Where the sign is the sign of the corresponding edge if and only if the edge agrees with the clockwise curl. eg $-a+b-c+b=0$.

2.13 Proposition *With the above choice of signs the sum of the crossing equations is the zero equation.*

Proof Observe that adjacent clockwise curls conflict in direction:



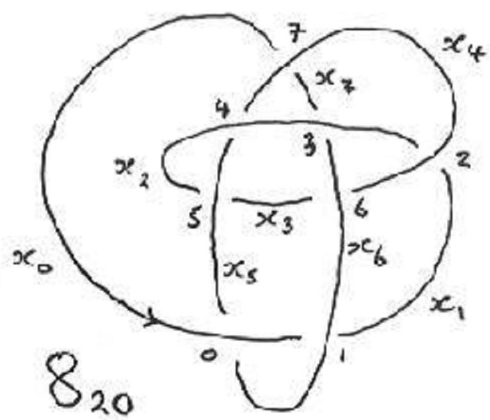
An edge participates with opposite signs in two adjacent equations and thus the contributions from that edge cancelled out. □

3 A systematic approach to colouring

Assume diagrams connected, reduced
and without closed curves

We look for integer solutions of eg:

$$\left.\begin{aligned} 2x_0 - x_5 - x_6 &= n b_0 \\ 2x_6 - x_0 - x_1 &= n b_1 \\ 2x_4 - x_1 - x_2 &= n b_2 \\ &\vdots \\ 2x_4 - x_0 - x_7 &= n b_7 \end{aligned}\right\} *$$



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for some $n \in \mathbb{Z}$ and (integer) vector b. Ie solve in integers $A_+ x = nb_+$ where :

$$b_+ = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_7 \end{bmatrix} \qquad A_+ = \overset{\text{arcs} \longrightarrow}{\text{crossings}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \end{bmatrix}}$$

3.1 Lemma *A_+ is a square matrix.*

Proof The number of crossings is equal to the number of arcs since if we chose an orientation then each arc ends at a crossing and this assignment clearly determines a bijection $arcs \to crossings$. □

We can forget any one equation since by 2.13 any equation is a signed sum of the others (we are free to choose the b's). By 2.3 we can assume any given x_j is 0. Hence it suffices to solve $Ax = nb$ where A is A_+ with i^{th} row and j^{th} column deleted—it does not matter which row and column is deleted—b is b_+ with i^{th} entry deleted. Assume the numbering chosen so there are $m+1$ arcs (and crossings). Put $x_0 = 0$ and forget the 0^{th} crossing—delete 0^{th} row and column of the matrix A_+.

Case i) $det(A) = 0$ Suppose $n = 0$ then since $\mathbb{Q}$ is a field there is a non-trivial solution, say $x_i = p_i/q_i \in \mathbb{Q}$, for $1 \le i \le m$. We get a colouring $(y_0, \ldots, y_m)$ in $\mathbb{Z}$ by taking $y_0 = 0, y_1 = x_1 q, y_2 = x_2 q, \ldots, y_m = x_m q$ where $q = q_1 \cdot \ldots \cdot q_m$. So 0 is a colouring number and hence any $n \neq 1$ is a colouring number. Watch out—first divide by n^p for maximum p to get a non trivial colouring *mod n*.

Case ii) $det(A) \neq 0$ Cramer's rule gives a unique solution in $\mathbb{Q}$:

$$x_k = \frac{\begin{vmatrix} a_{11} & \cdots & nb_1 & \cdots & a_{1m} \\ \vdots & & & & \vdots \\ a_{m1} & \cdots & nb_m & \cdots & a_{mm} \end{vmatrix}}{\begin{vmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}} \qquad 1 \le k \le m$$

$$= \frac{n\begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1m} \\ \vdots & & & & \\ a_{m1} & \cdots & b_m & \cdots & a_{mm} \end{vmatrix}}{det(A)}$$

Take $n = |det(A)|$ then we have a solution:

$$x_0 = 0, x_k = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & b_m & \cdots & a_{mm} \end{vmatrix}, 1 \le k \le m,$$

(columns 1, k, m)

of

$$Ax = \begin{cases} nb, & \text{if } \det A > 0 \\ n(-b) & \text{if } \det A < 0. \end{cases}$$

Take $(b_1, \ldots, b_m) = (1, 0, \ldots, 0)$. Then

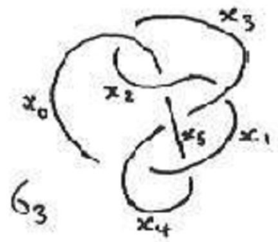
$x_0 = 0$, $x_k = (-1)^{k+1}$(the $1, k$ minor of A), for $1 \le k \le m$, and $\det A = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m$.

and notice that since $\det(A) \neq 0$ then some $x_i \neq 0$ and we have a non constant solution which is non constant *mod* n provided not all x_i are divisible by n.

Definition The *determinant* of a link L is $\det(L) = |\det(A)|$. We shall see later that $\det(L)$ is well defined.

Although Cramers's rule provides solutions for $n = \det(L)$ in many cases, it does not completely solve the problem.

3.2 Example *There is a convenient labelling for alternating links—arc x_i runs over the i^{th} crossing. We then get 2's running down the diagonal.*



	x_0	x_1	x_2	x_3	x_4	x_5
0	2	0	-1	-1	0	0
1	0	2	0	0	-1	-1
2	-1	0	2	0	0	-1
3	0	-1	-1	2	0	0
4	-1	-1	0	0	2	0
5	0	0	0	-1	-1	2

$$\det A = \begin{vmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{vmatrix}$$

$$x_0 = 0, \quad x_1 = \begin{vmatrix} 2 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{vmatrix}, \quad x_2 = - \begin{vmatrix} 0 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{vmatrix}, \quad x_3 \ldots$$

I do not know of an example where the method of 3.2 fails to find a colouring mod det(A). If you find one let me know.

Additional material on determinants.

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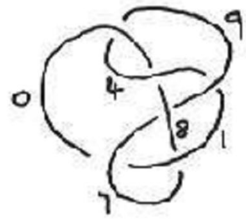
$x = (0, 14, 4, 9, 7, 8)$,
$\det(6_3) = 2x_1 + 0x_2 + 0x_3 + (-1)x_4 + (-1)x_5 = 28 - 7 - 8 = 13$,
$x = (0, 1, 4, 9, 7, 8) \quad mod \quad 13$.



3.3 Theorem *A number $n \neq 1$ is a colouring number for a link L if and only if n and* $\det L$ *have a common factor.*

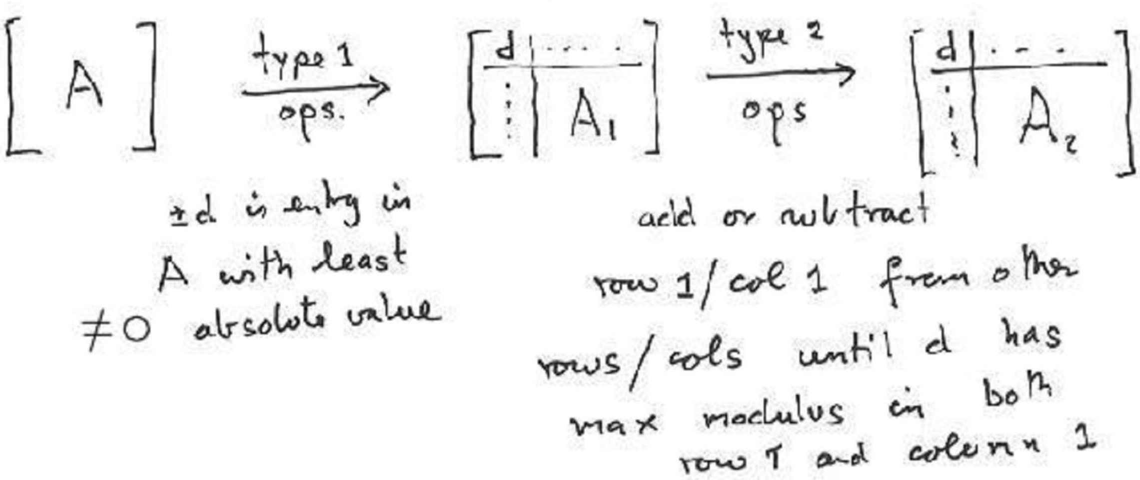
We will need:

3.4 Lemma *Given a square integer matrix A as above then there are integer matrices R, C, D so that R, C represent isomorphisms of $\mathbb{Z}^m$, D is diagonal,* $D = RAC$ and $\det D = \det A$.

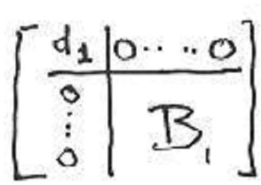
Proof of lemma Use row operations of type
1) $r_i \mapsto -r_j$, $r_j \mapsto r_i$ and 2) $r_i \mapsto r_i + ar_j, a \in \mathbb{Z}$

Similarly for column operations. These operations do not change determinant.

Step 1



Step 2 repeat step 1 until



now work on B_1 until

$$\left[\begin{array}{cc|c} d_1 & 0 & \\ 0 & d_2 & 0 \\ \hline 0 & & B_2 \end{array}\right]$$

continue until we get

$$D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}$$

□

Proof of Theorem Reduce *mod* n to get

$$\begin{array}{ccc} \mathbb{Z}_n^m & \xrightarrow{A_n} & \mathbb{Z}_n^m \\ {\scriptstyle C_n}\uparrow\cong & & \cong\downarrow{\scriptstyle R_n} \\ \mathbb{Z}_n^m & \xrightarrow{D_n} & \mathbb{Z}_n^m \end{array}$$

A colouring *mod* n corresponds to a non-trivial element in $\ker A_n$. But $C_n(\ker D_n) = \ker A_n$ and hence, since C_n and R_n are isomorphisms, to a non-trivial element in $\ker D_n$. Since $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{bmatrix}$ $\ker D_n \neq 0$ if and only if some $d_i\colon \mathbb{Z}_n \to \mathbb{Z}_n$ has $\ker d_i \neq 0$ ie $d_i k = 0$ *mod* n, $k \neq 0$ *mod* n. This happens if and only if $|d_i|$ and n have a common factor. So $\ker D_n \neq 0$ if and only if $\det L = |d| = |d_1 \ldots . d_n|$ and n have a common factor.

□

3.5 Remarks i) If we take it that 0 has a common factor with any n then the theorem includes the case $\det L = 0$ and then any $n \neq 1$ is a colouring number.

ii) Notice that if $\det L = 1$ then L has no colouring numbers.

iii) The colouring problem is now completely solved and the proof of the theorem could be used to find concrete solutions—keep track of the column operations and apply C_n to an element in $\ker D_n$.

iv) The proof of the theorem is based on the use of a reduced connected diagram without closed curves. The definition of $\det L$ can be extended by using arbitrary diagrams as follows.

An example for 3.4.
See also algebra2 Theorem 3.2.2 and Hans J Zassenhaus, The theory of groups, Theorem 18. Here is a link to the library copy.

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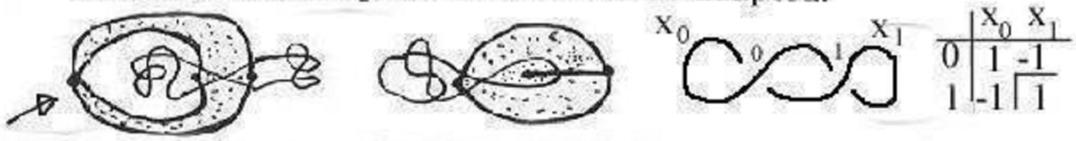
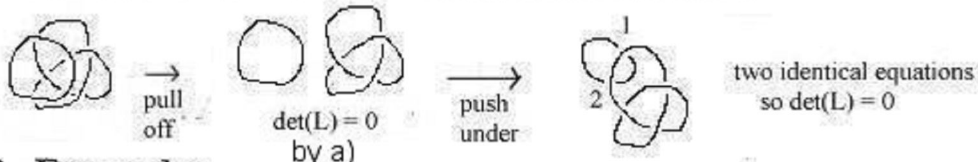
a) If the diagram is not connected the equations coming from any one piece are dependent so 2.13 holds.
The equations from any other piece are also dependent so det(L)=0

b) In case the diagram is not reduced the notion of 'quadrilateral' can be stretched and the proof of 2.13 can be adapted.



Two corners identified   Two edges identified   unknot   det = 1

c) If there are closed curves and more than one component then the matrix is no longer square—more crossings than arcs, but consider the isotopies:

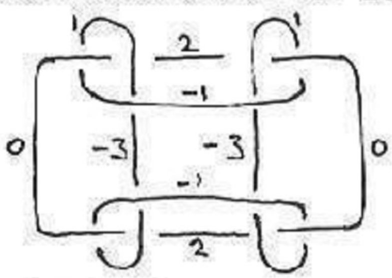


3.6 Examples

i) See example sheet, question number 13. There are links which cannot be distinguished by colouring numbers but have distinct determinants.

det = 4     det = 8

ii) If a link is splittable then $\det L = 0$. The converse is false:

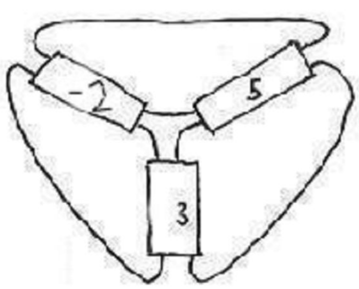


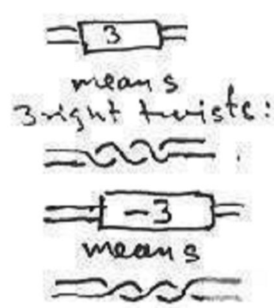
A solution in $\mathbb{Z}$ so $\det L = 0$.

iii) There exist knots having no colouring numbers; $\det L = 1$. See example sheet, question number 9. Here is another:

A pretzel knot

P(5,3,-2) = 10_{124}





More on P(5,3,-2)

Definition let L be a link with diagram D. Define the colouring group $Col(L)$ to be the abelian group with generators all arcs except one and relations the crossing equations:

$$Col(L) = < a, b, c, \ldots | 2b = a + c, \ldots >$$

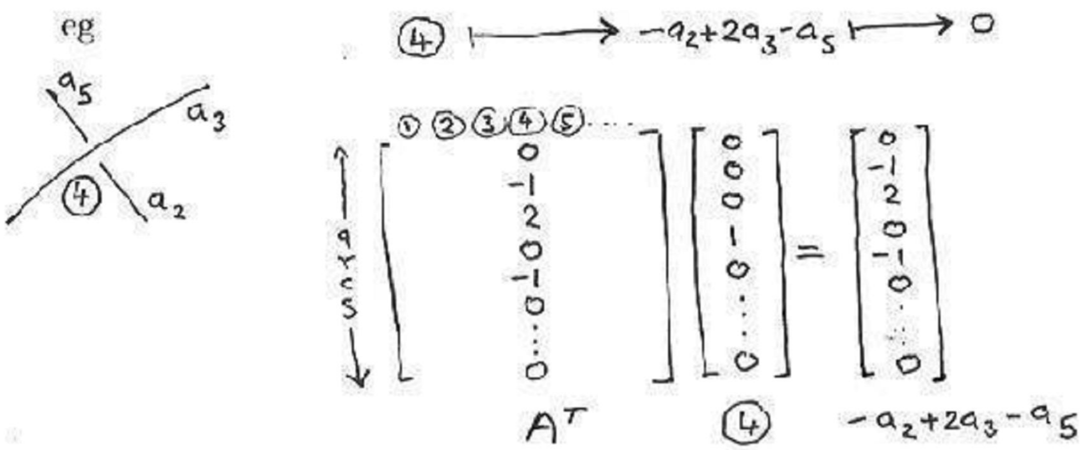
where on the right the missing arc appears as 0. Assuming no closed loops this group can be identified with the quotient group $\mathbb{Z}^m / A^T \mathbb{Z}^m$ where A^T is the transpose of the matrix A we used earlier. We have:

$$\begin{Bmatrix}\text{solutions}\\ \text{mod n}\end{Bmatrix} \longrightarrow \underset{\text{basis of arcs}}{\mathbb{Z}_n^m} \xrightarrow{A} \underset{\text{basis of crossings}}{\mathbb{Z}_n^m}$$

$$\underset{\text{basis of crossings}}{\mathbb{Z}^m} \xrightarrow{A^T} \underset{\text{basis of arcs}}{\mathbb{Z}^m} \longrightarrow Col(L)$$

Note 1 arc and 1 crossing disregarded

eg

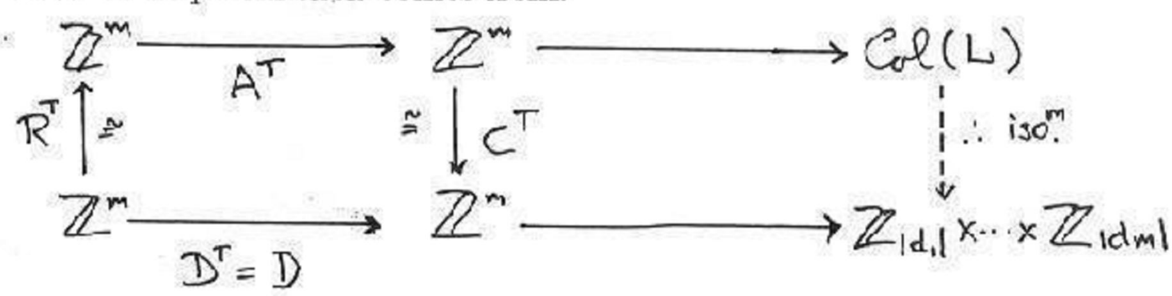


3.7 Proposition

With notation as above

$$Col(L) \cong \mathbb{Z}_{|d_1|} \times \ldots \times \mathbb{Z}_{|d_m|}$$

Proof If we diagonalise A^T we get the same result as diagonalising A. The isomorphism then comes from:



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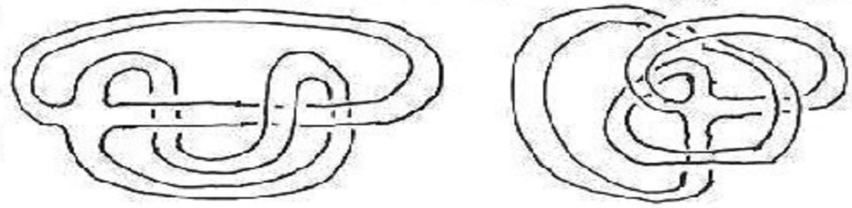
Other lectures 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30

3.8 Remarks

i) It follows that if $\det L \neq 0$ then $\det L$ is the order of the colouring group, ie $|d_1|\ldots|d_m|$. By analysing the effect of the Reidemeister moves it can be shown as in the proof of 2.2 that $Col(L)$ and hence $\det L$ is well a defined isotopy invariant.

ii) The group is stronger than the determinant. For example $\det(9_{46}) = \det(6_1) = 9$ but $Col(9_{46}) = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $Col(6_1) = \mathbb{Z}_9$. We omit the proof.



3.9 Proposition *Given a link L with $Col(L)$ as in 3.7 above then the number of distinct colourings in $\mathbb{Z}_n$* , $n \neq 0$,

(with 0 on a fixed arc and counting the constant colouring *) of L is*

$$gcd(|d_1|, n) \times \cdots \times gcd(|d_m|, n)$$

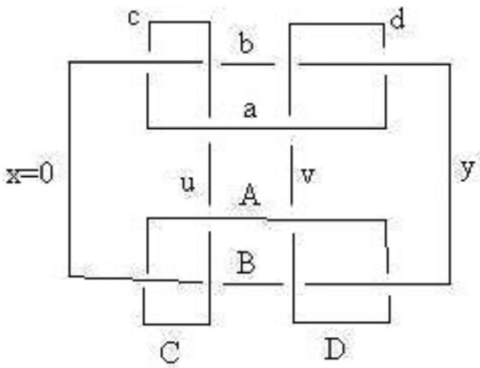
Proof A colouring in $\mathbb{Z}_n$ is an assignment of elements of $\mathbb{Z}_n$ to the arcs of a diagram which satisfies the crossing equations. This is equivalent to a homomorphism $c: Col(L) \to \mathbb{Z}_n$. These homomorphisms form a group $Hom(Col(L), \mathbb{Z}_n)$. The order of this group is the number of distinct colourings (with 0 on a fixed arc). Since $Col(L) \cong \mathbb{Z}_{|d_1|} \times \ldots \times \mathbb{Z}_{|d_m|}$ the result follows from the fact that $Hom(\mathbb{Z}_a, \mathbb{Z}_b) \cong \mathbb{Z}_{gcd(a,b)}$ and $Hom(\mathbb{Z}_a \times \mathbb{Z}_b, \mathbb{Z}_c) \cong Hom(\mathbb{Z}_a, \mathbb{Z}_c) \times Hom(\mathbb{Z}_b, \mathbb{Z}_c)$. □

Example In $\mathbb{Z}_3$ the knot 9_{46} has 9 colourings but 6_1 has only 3.

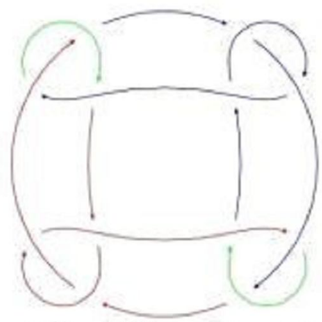
A non splittable link with comp(L) = 2, det=0, Col(L) = $Z \times Z_3 \times Z_3$

So it has 27 3-colourings whereas the 2 component unlink has only 3

(0 on a fixed arc and counting in the constant colouring) This is 3.6 ii again



Thanks to Roger Fenn for this pretty picture of a 3-colouring



A =

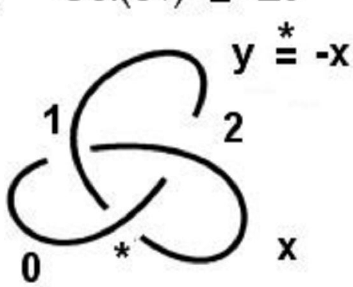
y	a	b	c	d	A	B	C	D	u	v
0	0	0	0	0	0	-1	2	0	0	0
-1	0	0	0	0	0	-1	0	2	0	0
2	0	0	0	0	-1	0	0	-1	0	0
0	0	0	0	0	2	0	-1	0	-1	0
0	0	0	0	0	2	0	0	-1	0	-1
0	-1	0	-1	0	0	0	0	0	0	0
0	0	-1	2	0	0	0	0	0	0	0
-1	0	-1	0	2	0	0	0	0	0	0
2	-1	0	0	-1	0	0	0	0	0	0
0	2	0	-1	0	0	0	0	0	-1	0
0	2	0	0	-1	0	0	0	0	0	-1

After diagonalising get two 3s a 0 and eight 1s down the diagonal

Remark 3.10

The computations of 2.4 could be viewed as computing Col(L).

eg Col(3_1) ≅ Z_3 generator x relation 3x=0

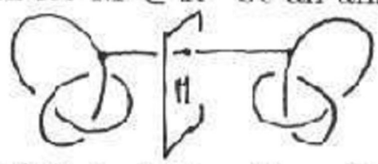


< x,y | y $\overset{*}{=}$ -x, 2y $\overset{1}{=}$ x, 2x $\overset{2}{=}$ y>

= < x | 3x=0 >

4 Mirrors and knot coding

Let L be a link and let $M \subset \mathbb{R}^3$ be an affine plane.



The *mirror* of L, $m(L)$, is obtained by reflection in M (the link is allowed to meet M).

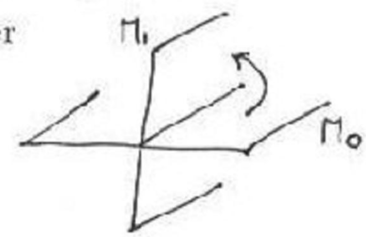
4.1 Proposition The operation *mirror* acts on isotopy classes and does not depend on the choice of mirror

Proof Let $m_\bullet$ denote reflection in $M_\bullet$. We show that if M_0, M_1 are mirrors and L_0, L_1 are isotopic links then $m_0 L_0$ is isotopic to $m_1 L_1$.

Step 1 Given M, L_0 isotopic to L_1 implies mL_0 isotopic to mL_1, since if L_t is an isotopic of L_0 to L_1, then mL_t is an isotopy of mL_0 to mL_1.

Step 2 We show $m_0 L_1$ isotopic to $m_1 L_1$ for mirrors M_0 and M_1. Choose a homotopy of isometries $f_t\colon \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $f_0 = identity$, $f_1 M_0 = M_1$ so that:

Either



f_t rotates about $M_0 \cap M_1$

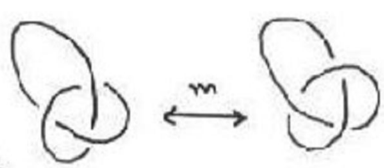
or

f_t translates

Let $M_t = f_t M_0$ be the mirror at time t. Then if $m_t(L)$ denotes reflection of L in M_t we have $m_0(L)$ isotopic to $m_1(L)$ through $m_t(L)$. Steps 1 and 2 together give the result:

$$m_0 L_0 \sim m_0 L_1 \sim m_1 L_1$$

4.2 Proposition *A diagram for mL is obtained from a diagram for L by changing all crossings:*



right and left trefoils

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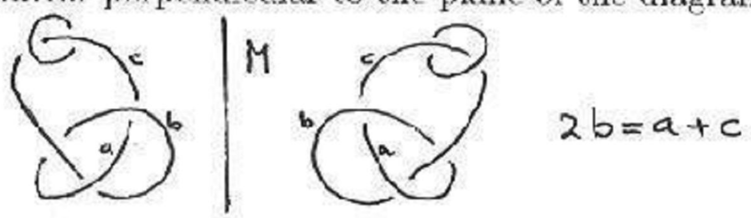
Proof The knot may be obtained from the diagram by lifting over crossings a little and lowering under crossings a little. Then choose M to be the plane of the diagram. □

Definition A link is *achiral (amphichiral)* if it is isotopic to its mirror image, otherwise it is *chiral.*

Remark The knot table only shows one of each (possible) pair K, mK. The achirals are 4_1, 6_3, 8_3, 8_9, 8_{12}, 8_{17}, 8_{18}. It is hard to show 3_1 is chiral.

4.3 Proposition *Colouring numbers,* $\det(L)$, $Col(L)$, *cannot detect chirality.*

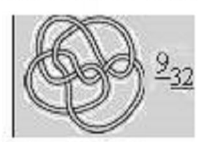
Proof Use a mirror perpendicular to the plane of the diagram:



We get the same result in both cases. □

Definition Let K be an oriented knot an let $r(K)$ denote K with orientation reversed. If K and $r(K)$ are isotopic (as oriented knots) then K is *invertible.*

Remark There is only one non-invertible knot in the table: 8_{17}. This is hard to prove. If $K = 9_{32}$ with an orientation choice then K, mK, rK, and $rmK = mrK$ are all distinct (oriented) knots.



Knot coding

Recall that a shadow is a link diagram without crossing information.

4.4 Proposition *A shadow with one `component'*

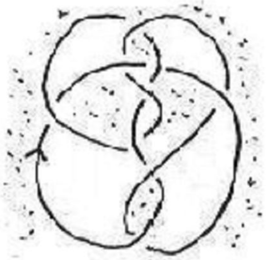
represents a unique (up to mirrors) unoriented alternating knot.

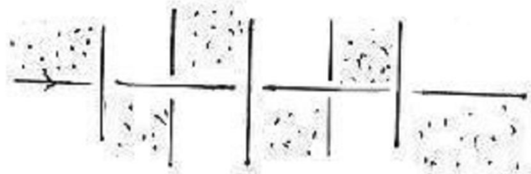
Proof Walk round the shadow introducing crossing information—under, over, under, over, under, ... To see that this works

choose a chess boarding with

black on the left as we start to walk and observe that when black / white is on the left then we go under/over next.

9_{32}



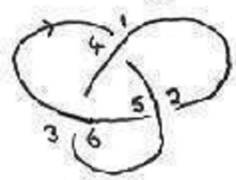


□

Remark Only the last three in the table are not alternating—8_{19}, 8_{20}, 8_{21}.

Given a shadow, walk round it numbering crossings consecutively. We now have a function $f\colon \{odd \quad numbers\} \to \{even \quad numbers\}$.
undercrossings overcrossings
(if alternating)

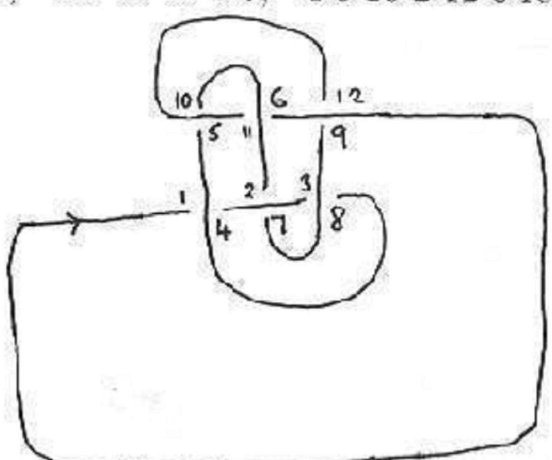
The sequence $f(1), f(3), f(5), \ldots$ determines the shadow (uniquely in S^2)‡ and hence determines an alternating knot up to mirrors. eg 3_1 is represented by 4 6 2.



Different shadows of the same knot will give different sequences. But the sequence least in lexicographical order uniquely represents the knot. For non-alternating knots insert minus signs where crossings are 'wrong' **ie undercrossing labelled even.**

Examples The knots 6_1, 6_2, 6_3, are given by
4 8 12 10 2 6, 4 8 10 12 2 6, 4 8 10 2 12 6 respectively.

6_3



‡ unique provided no proper subinterval J of (1,2,3,...,2n), where n is the number of crossings satisfies f{odd nums in J} = {even nums in J}

Remark. Given n consider the set of knots which have a diagram with n crossings but not fewer crossings. Then for n = 3,4,5,6,7 colouring numbers distinguish the knots from each other. But 8_2 and 8_3 have the same colouring group Z_{17}. In the next section we will have an invariant which distinguishes these two knots.

8_17 is achiral. A table of knots (up to 10 crossings) and their codes. Some diagrams drawn from their codes. See what Tait had to say about 4.4 and Celtic knots.

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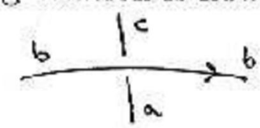
5 The Alexander polynomial

We have studied colouring in $\mathbb{Z}$ mod n. We now generalise: replace $\mathbb{Z}$ by $\mathbb{Z}[t, t^{-1}]$ the ring of Laurent polynomials (one variable t, negative powers allowed) and attempt to 'colour' oriented links (with polynomials) mod a polynomial. The crossing relation is now:

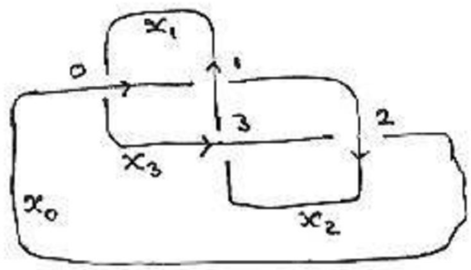


$$c - ta = b - tb \text{ or } c = ta + (1 - t)b.$$

Notice the direction on the undercrossings is not used, and if we put $t = -1$ the old crossing relation is recovered. We get a determinant $\Delta_L(t)$ as before and find colouring in $\mathbb{Z}[t, t^{-1}]$ mod $\Delta_L(t)$. The polynomial $\Delta_L(t)$ is the *Alexander polynomial* of L.

5.1 Example

4_1



For ease of reading off the equations place the direction arrows on the overcrossings.

	x_0	x_1	x_2	x_3
0	$1-t$	-1	0	t
1	-1	$1-t$	t	0
2	-1	0	$1-t$	t
3	0	-1	t	$1-t$

$$x_0 = 0$$

$$x_1 = \begin{vmatrix} 1-t & t \\ t & 1-t \end{vmatrix} = 1 - 2t$$

$$x_2 = -\begin{vmatrix} 0 & t \\ -1 & 1-t \end{vmatrix} = -t, \quad x_3 = \begin{vmatrix} 0 & 1-t \\ -1 & t \end{vmatrix} = 1 - t$$

The determinant of the (0,0)-minor is:

$$\begin{aligned}\Delta_{4_1}(t) &= (1-t)x_1 + tx_2 + 0(x_3)\\ &= (1-t)(1-2t) - t^2\\ &= 1 - 3t + t^2\end{aligned}$$

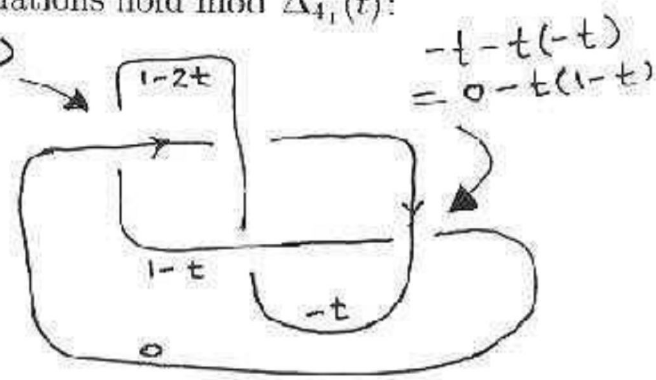
NB The Alexander polynomial $\Delta_L(t)$ is well defined only up to multiplication by $\pm t^n$, n in Z. This is because to get 2.13 to work we have to multiply equations by $\pm$ powers of t. We write

$$\begin{aligned}\Delta_{4_1}(t) &\doteq 1 - 3t + t^2\\ &\doteq t^{-1} - 3 + t\end{aligned}$$

for example. Check that relations hold mod $\Delta_{4_1}(t)$:

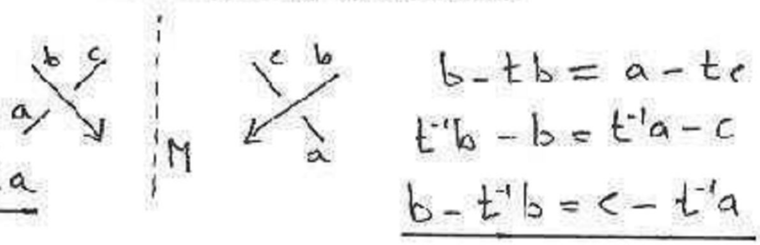
$0 - t0 = (1-2t) - t(1-t)$

$0 = t^2 - 3t + 1$



5.2 Proposition
i) $|\Delta_L(-1)| = \det(L)$.
ii) $\Delta_{rL}(t^{-1}) \doteq \Delta_L(t) \doteq \Delta_{mL}(t^{-1})$.

Proof i) follows from definitions. For ii) consider:



This shows $\Delta_L(t) \doteq \Delta_{mL}(t^{-1})$. Similarly $\Delta_L(t) \doteq \Delta_{rL}(t^{-1})$. □

5.3 Remark In fact $\Delta_L(t) \doteq \Delta_L(t^{-1})$ as well for knots. ☹

5.4 Proposition $\Delta_L(t) = 0$ *if* L *is splittable.*

Additional material on the Alexander polynomial.
A Maple worksheet for calculating Alexander polynomials and determinants of knots and links.
Here is Alexander's paper on the polynomial:

Topological Invariants of Knots and Links, J. W. Alexander, Transactions of the American Mathematical Society, Vol. 30, No. 2. (Apr., 1928), pp. 275-306.

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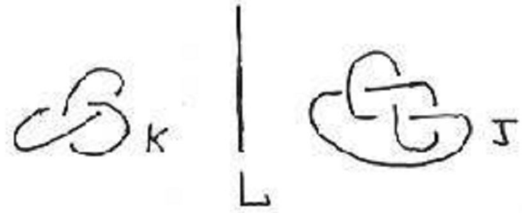
Proof



Let A_+, B_+ be the full matrix of relations for K, J respectively Then the full matrix of relations for L is

$$C_+ = \left[\begin{array}{c|c} A_+ & 0 \\ \hline 0 & B_+ \end{array}\right]$$

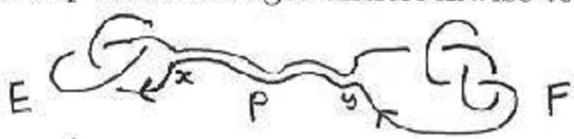
and $\det A_+ = \det B_+ = 0$ (from the analogue of 2.13). Then

$$C = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B_+ \end{array}\right]$$

so $\det C = \det A . \det B_+$ and $\Delta_L(t) = \Delta_K(t).0 = 0$. □

Oriented knot sums

For oriented knots K, L define $K\#L$ by choosing representing diagrams E in $x < 0$ and F in $x > 0$ respectively so that each has a clockwise outside arc (an R_1 move changes anticlockwise to clockwise)

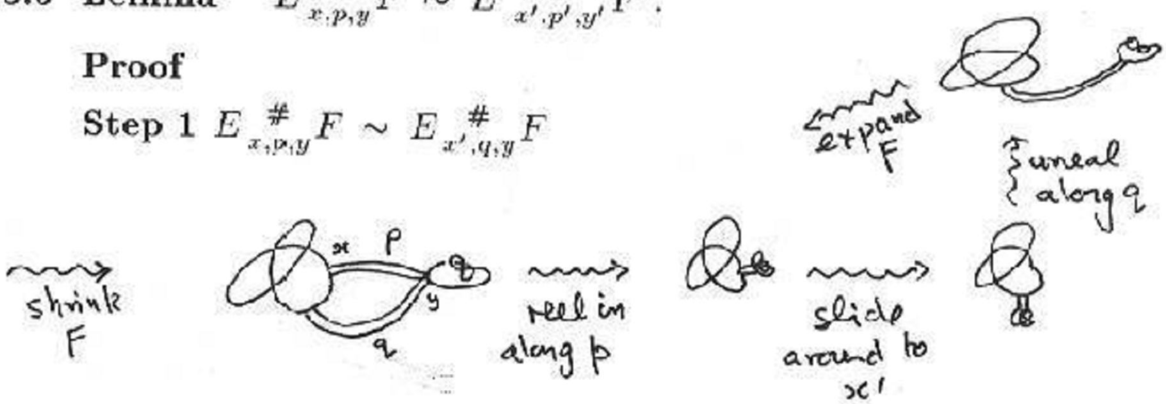


Define $E \mathop{\#}_{x,p,y} F$ by choosing a path from $x \in E$ to $y \in F$, then doubling the path and connecting as shown. Suppose we make different choices and get $E' \mathop{\#}_{x',p',y'} F'$. Let $\sim$ denote isotopy of links or diagrams.

5.5 Lemma $E \mathop{\#}_{x,p,y} F \sim E' \mathop{\#}_{x',p',y'} F'$.

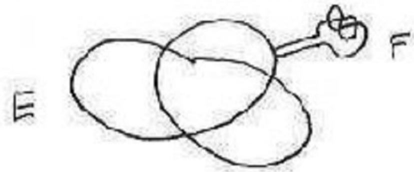
Proof

Step 1 $E \mathop{\#}_{x,p,y} F \sim E \mathop{\#}_{x',q,y} F$



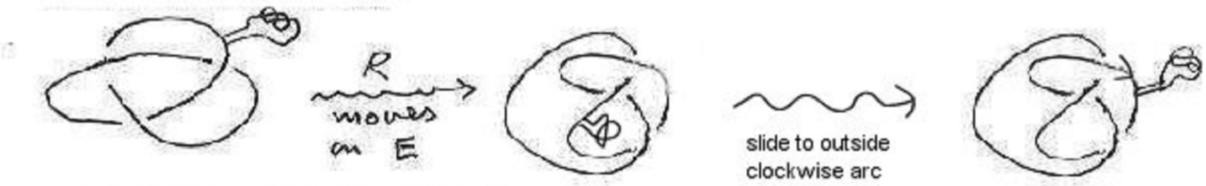
Similarly $E \mathop{\#}_{x',q,y} F \sim E \mathop{\#}_{x',p',y'} F$. This shows for fixed E and F $E\#F$

is well defined up to (diagram) isotopy and is represented by:



Step 2

Suppose $E \sim E'$ then $E\#F \sim E'\#F$ since (the small) F can be carried along by the isotopy (Reidemeister moves)



Similarly $E'\#F \sim E'\#F'$ □

5.6 Theorem

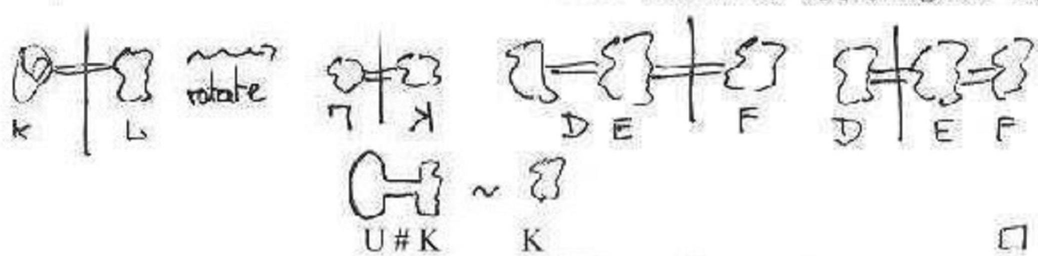
1) $K\#L$ is well defined up to isotopy and depends only on the isotopy classes of K and L

2) $K\#L \sim L\#K$

3) $K\#(L\#M) \sim (K\#L)\#M$

4) $U\#K \sim K$ where U is the unknot

Proof 1) follows from the lemma. The rest follow by applying an R_0 move.



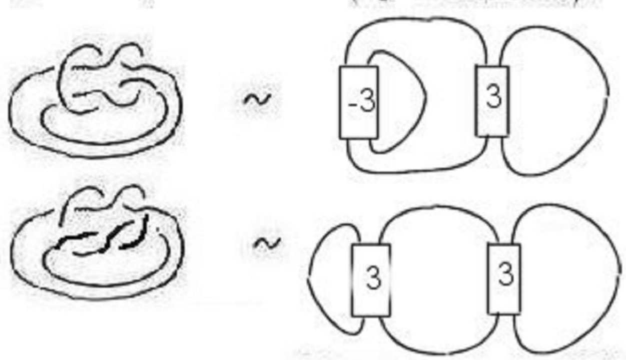
□

Definition A knot K is *prime* if $K \not\sim L\#M$ for non-trivial L, M. The knot tables only show prime knots (up to mirrors).

Examples

reef =
trefoil # mirror trefoil

granny =
trefoil # trefoil



code
4 6 2 10 12 8
Same in each case

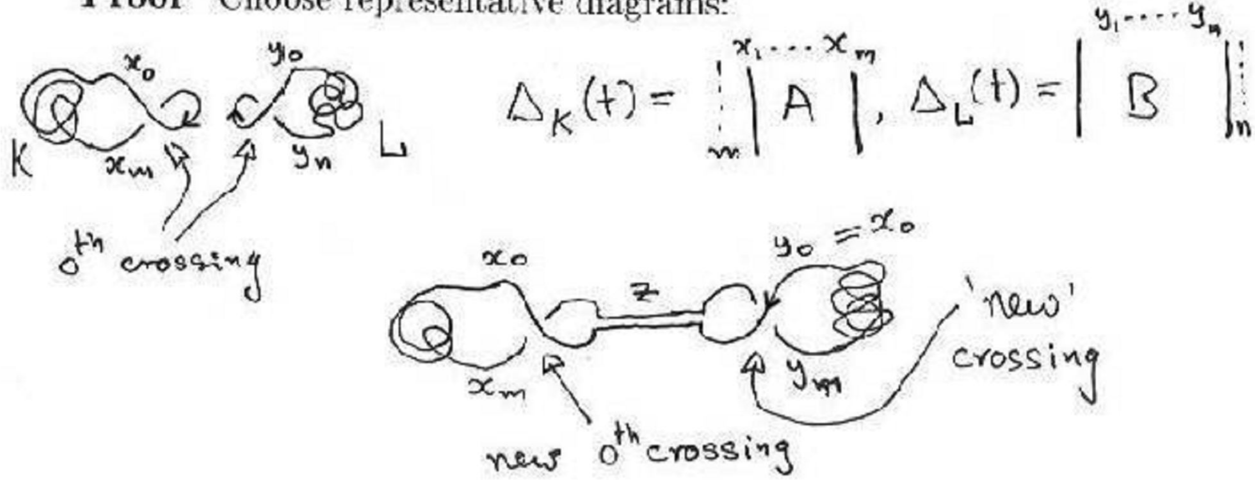
Trefoil is 3_1

It is not easy to prove these two are different. Their Alexander polynomial is

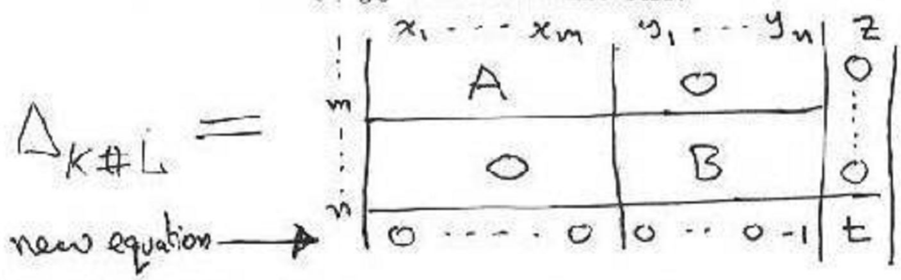
$$(1-t+t^2)^2 = 1 - 2t + 3t^2 - 2t^3 + t^4 \doteq t^{-2} - 2t^{-1} + 3 - 2t + t^2$$

5.7 Theorem $\Delta_{K\#L}(t) \doteq \Delta_K(t).\Delta_L(t)$ for knots K, L.

Proof Choose representative diagrams:



Get a new arc z. Old arcs x_0, y_0 have become one.



Expand from the right hand column to get the result. □

The reef knot is a sum of two trefoils, the Dufallandy knot is a sum of four copies of a 9-crossing knot, and here is a Celtic beast which has swallowed a sum of eight knots.

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6 Bridge number Plats and Braids

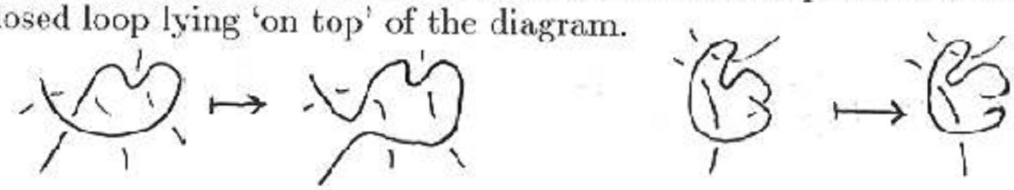
Definition The *bridge number* of a knot diagram D is the number of overcrossing arcs—*bridges*.

Definition The *bridge number*, $b(K)$, of a knot K is the minimum number of bridges that can occur in a diagram for K; define $b(K) = 1$ if K is the unknot.

6.1 Remark All knots in the table are 2-bridge knots, ie $b(K) = 2$, except for 8_5, 8_{10} and 8_{15}—8_{21} which are 3-bridge knots. 3-bridge knots are still not fully classified.

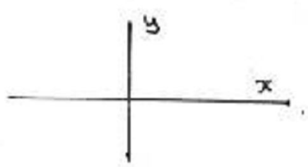
6.2 Proposition *For a knot K, $b(K) = 1$ implies K is the unknot.*

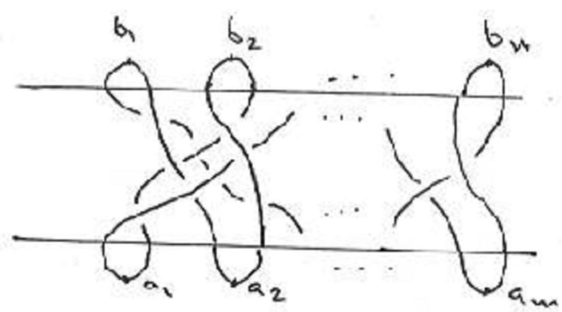
Proof Assume K is not the unknot and choose a diagram for K with the minimal number of crossings such that it has only one bridge. The bridge must start somewhere and returned to that point thus making a closed loop lying 'on top' of the diagram.



The closed loop bounds a disc by the Schöenflies Theorem and so can be turned over thus reducing the number of crossings. This is a contradiction. □

Definition A $2m$-*plat* (diagram):

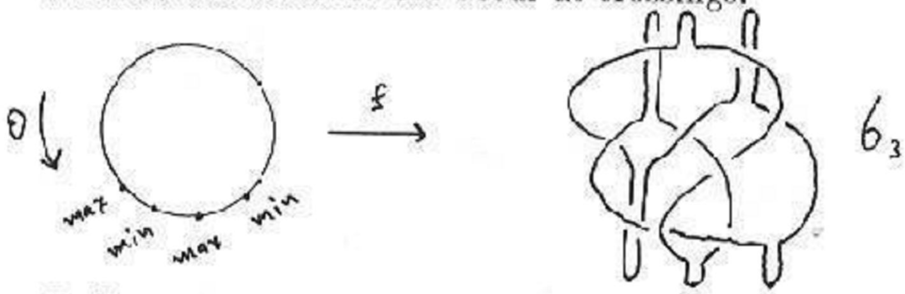




$\frac{dy}{d\theta} = 0$ only at a_i, b_i $1 \leq i \leq m$.
$a_1, \ldots, a_m$ are points on the diagram with minimum y value.
$b_1, \ldots, b_m$ are points on the diagram with maximum y value.
All the a_i's have the same y co-ordinate, and the same for the b_i's.

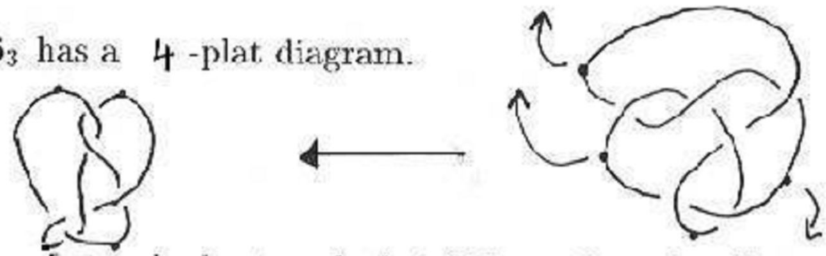
6.3 Proposition *Every knot has a plat diagram.*

Proof Choose a diagram so that $\frac{dy}{d\theta} = 0$ only at local maxima and local minima and these do not occur at crossings.



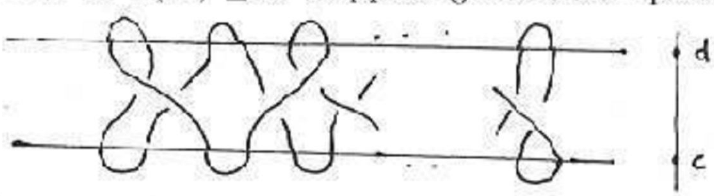
Pull up the maxima and pull down the minima as indicated. Note the number of maxima is equal to the number of minima since they occur alternately on the domain. □

Example 6_3 has a 4 -plat diagram.

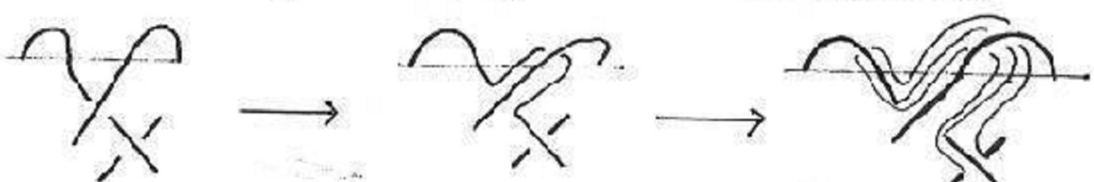


6.4 Proposition *Let m be least such that K has a 2m-plat diagram then* $b(K) = m$.

Proof **Step 1.** $b(K) \leq m$ Suppose given a 2 m-plat diagram for K:



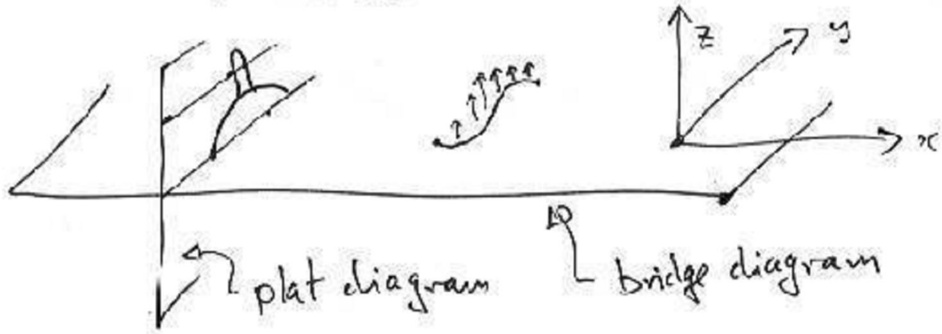
The idea is to turn the top hoops into bridges. The proof is by induction on the number of crossings with y co-ordinate in $[c, d]$. No crossings implies K is the unknot by the Schöenflies Theorem. To reduce the number divert the highest crossing as shown, repeat with the next highest, etc



We end up with no crossings with height in $[c, d]$.

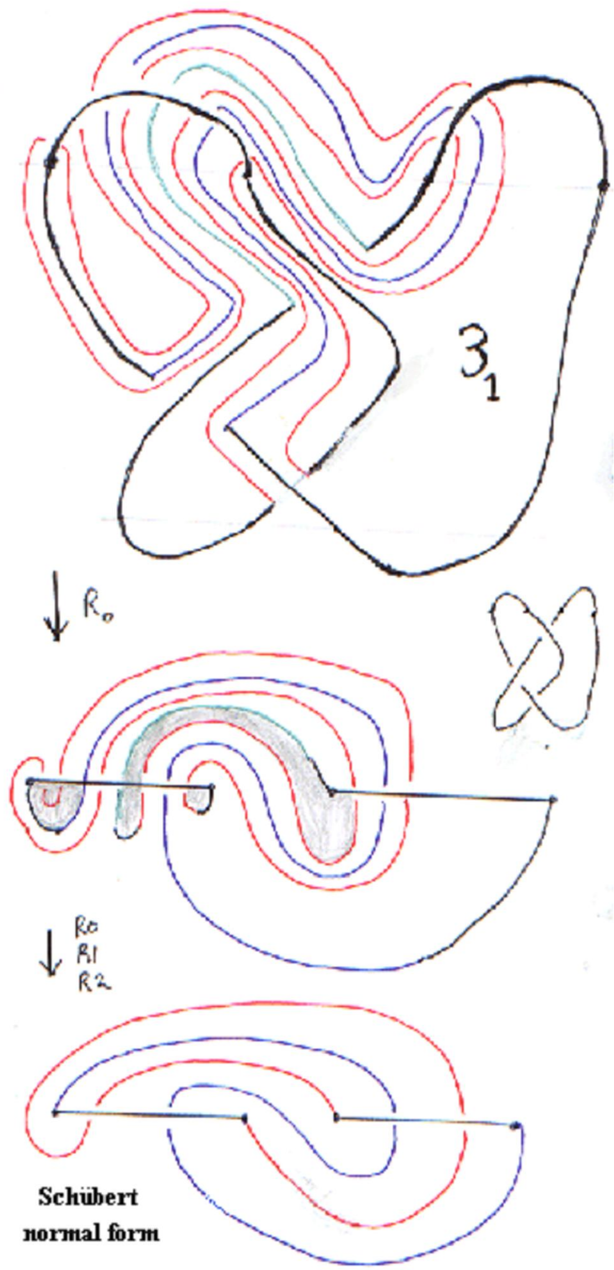
The number of bridges is now at most **the number of top hoops.**

Step 2. $b(K) \geq m$. Suppose given a diagram with $b(K)$ bridges. Construct a knot with exactly $b(K)$ maxima by raising the bridges (in the z direction—ends fixed). Project onto the y, z-plane to get a diagram with b(K) maxima (and b(K) minima after lowering the underpasses). The result follows as in the proof of 6.3.



□

6.5 Remark The process described in the proof that $b(K) \leq m$ produces a knot with many crossings. For example 3_1 gets 11 crossings. After flattening the two bridges and applying some R_0, R_1, and R_2 moves we get to the Schubert *normal form* for 2-bridge knots.



Two straight intervals. Each remaining arc runs from one interval to the other. All crossings are on the intervals.

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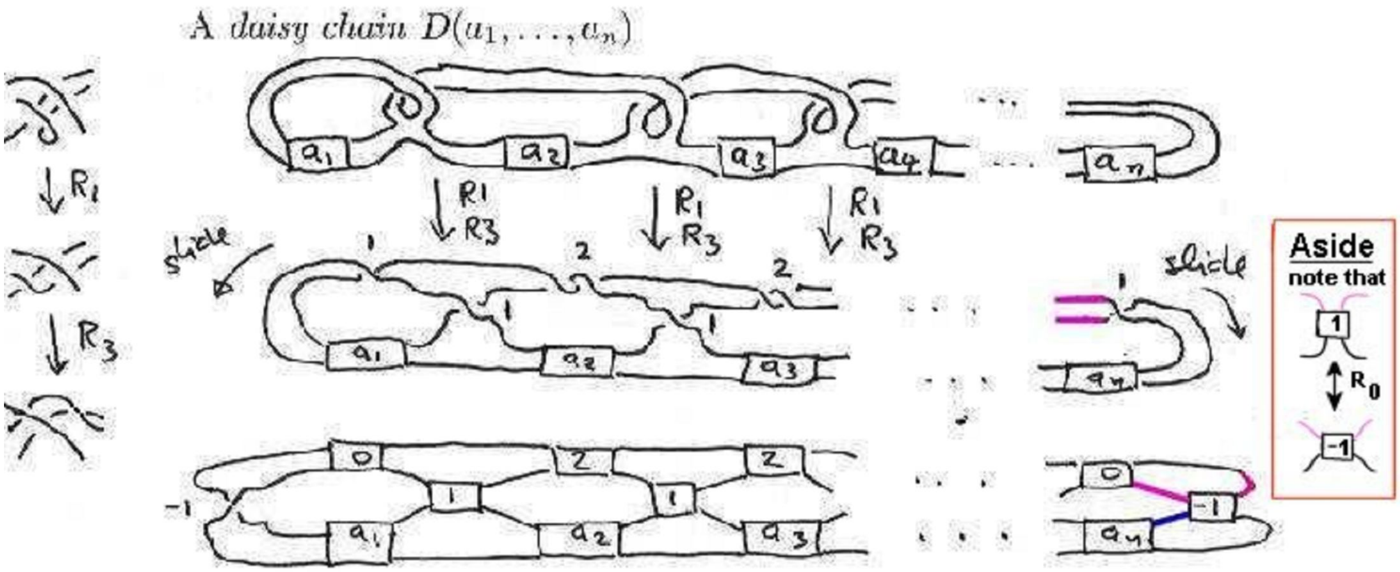
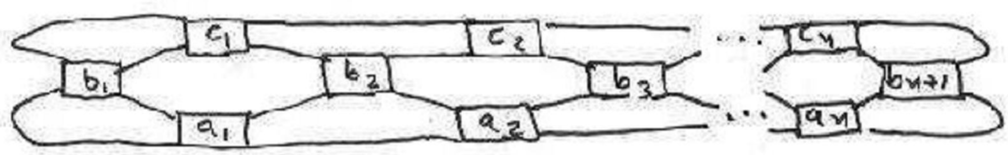
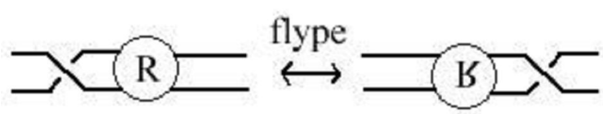


This shows that if a link has a daisy chain diagram then it has a 4-plat diagram. Here is a typical 4-plat diagram (why?):

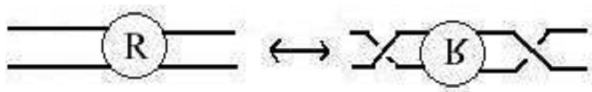


6.6 Theorem *Any 4-plat has a diagram as above with all ci's zero.*

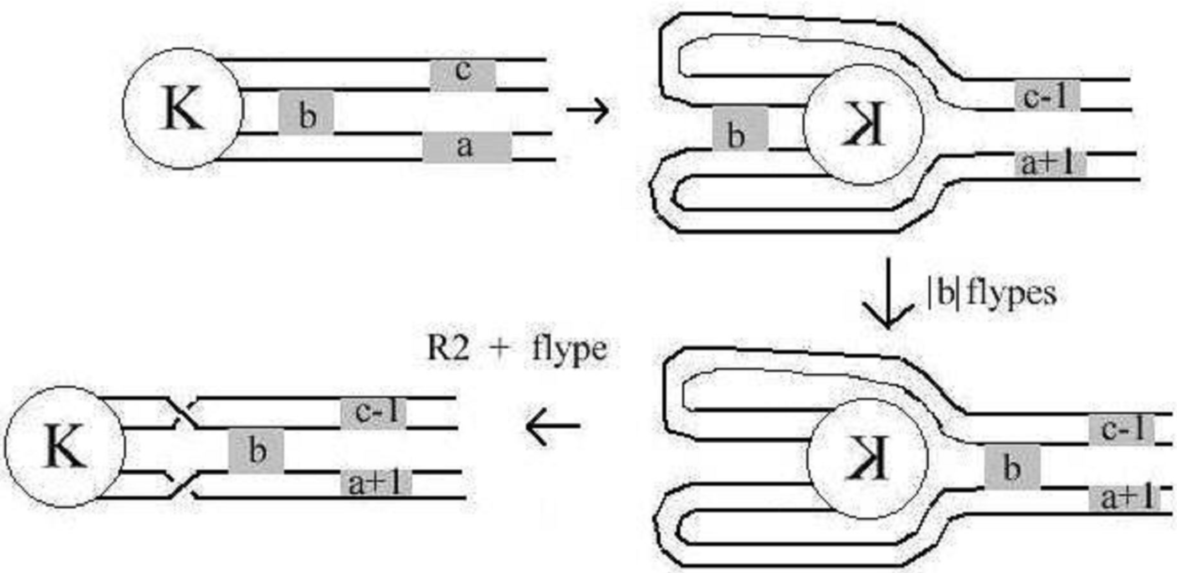
Proof The idea is to turn *c*'s into *a*'s by **flyping**. A **flype** moves a crossing across a part of a diagram at the expense of turning the part over:



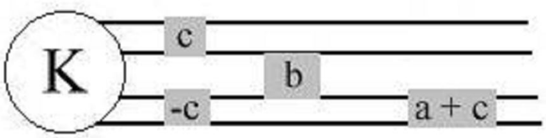
Notice that if we introduce two crossings using R2 then move one over with a flype we get:



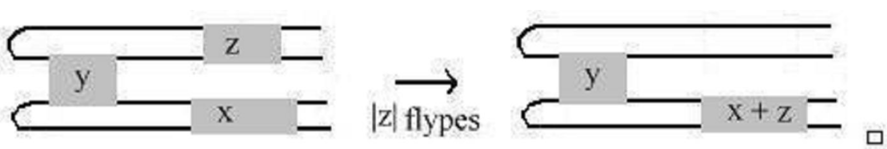
Suppose $c > 0$ and consider the sequence of flypes:



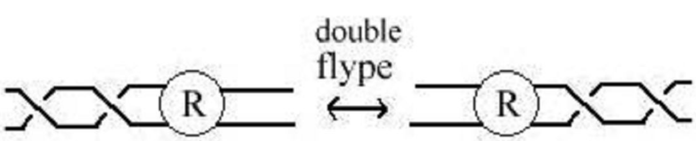
Repeat c times to get:



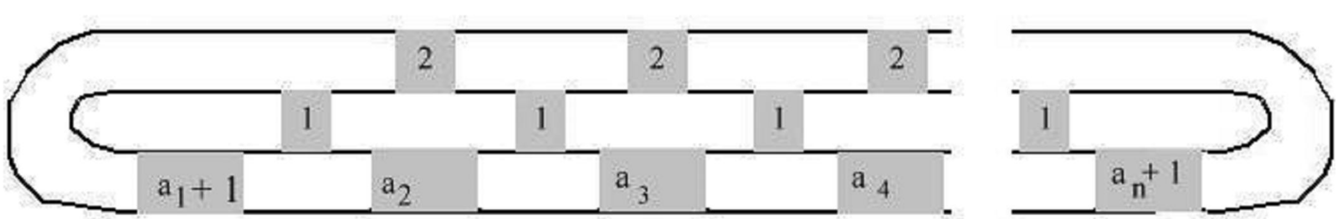
Similarly if $c < 0$. If b is odd K will have been rotated through pi about an horizontal axis. Use this procedure to eliminate c_n , c_{n-1} , ... , c_2 in turn. Then finally:



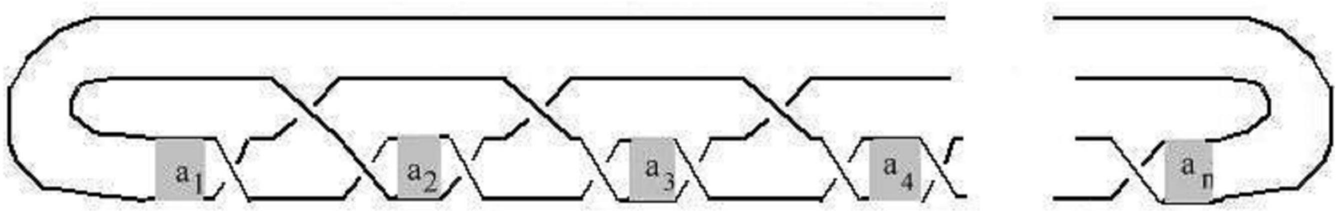
Notice that a double flype leaves R undisturbed:



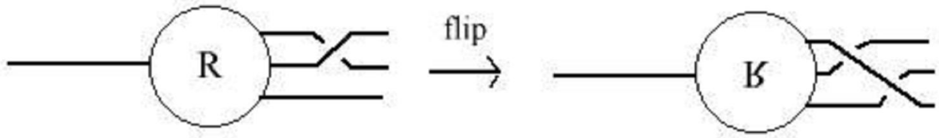
We can be more efficient in unwrapping a daisy chain. For example:



After n - 2 double flypes we have:

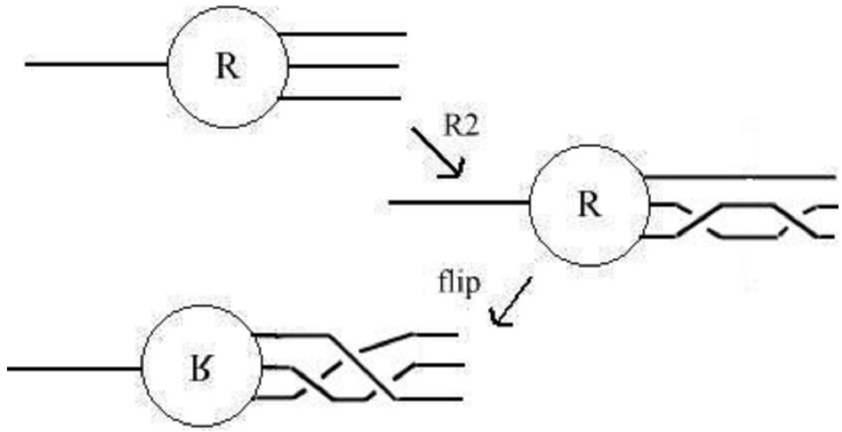


We now use **flips** to eliminate the unshaded crossings. Here is a **flip**:

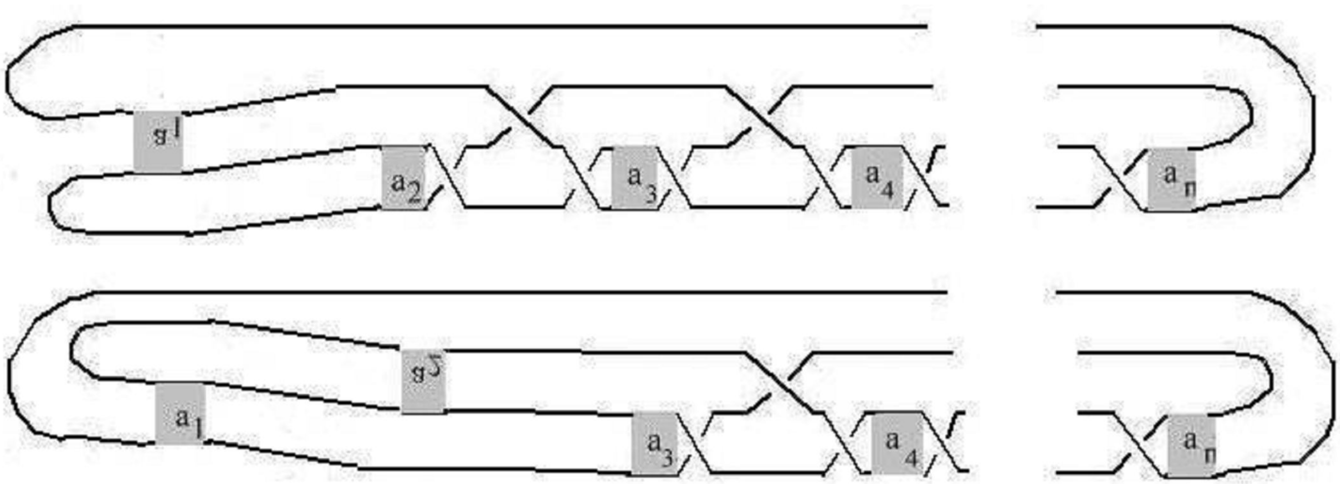


Think of this as turning R over to change the sign of a crossing between middle and outer string at the expense of adding a crossing between middle and other outer string.

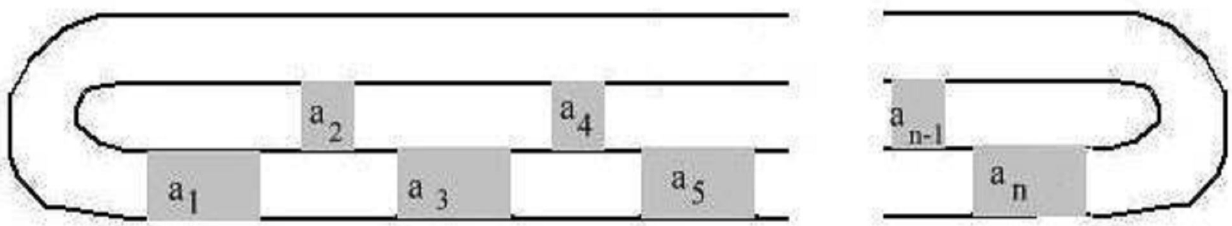
Combine a flip with R2 to get 3 new crossings all of the same sign as follows:



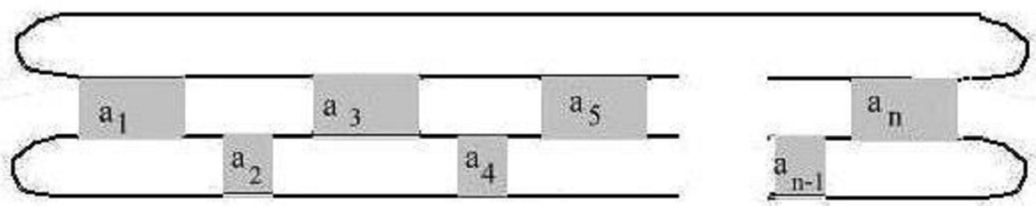
Apply R2 flips:



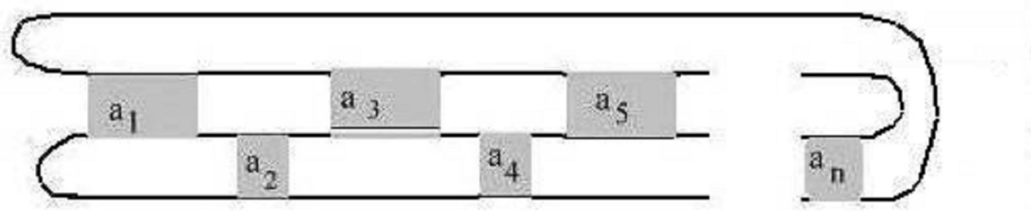
etc.
If n odd get:



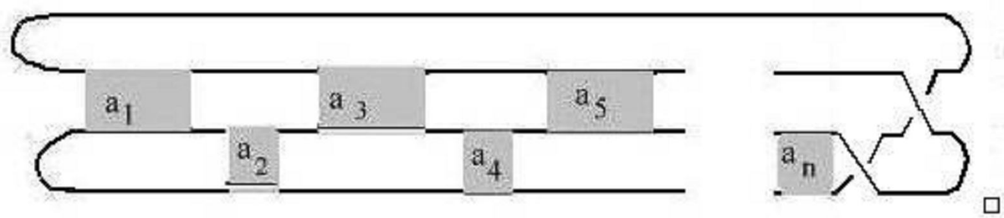
which is isotopic to



and if n even get:



which is isotopic to:

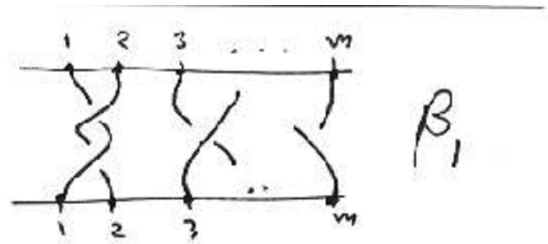


□

Remark It follows that the classes of knots defined by Daisy chains, 4-plats and 2-bridge knots are identical.

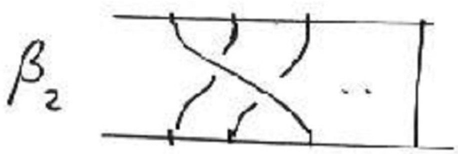
See the first fourteen prime knots added together. Notice that they are all 2-bridge knots.

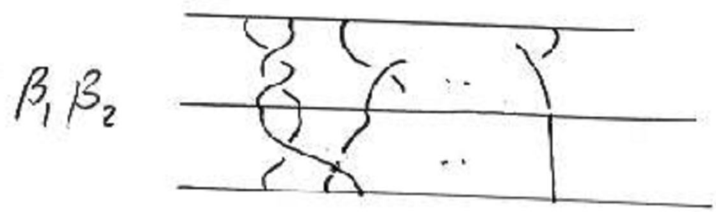
Definition An m-braid (diagram)



$\frac{dy}{d\theta}$ never zero, top and bottom points fixed as shown

Composition of braids:



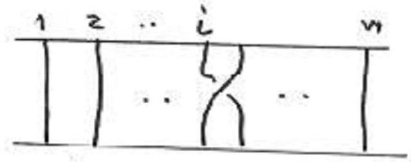


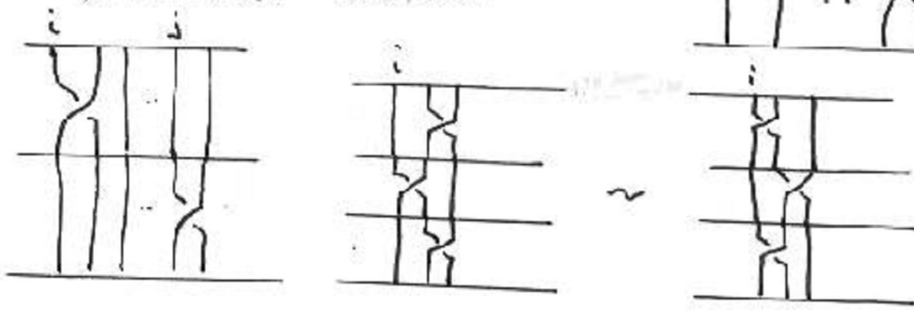
Regard two braids as equal if they differ only by isotopy through braids.

6.7 Theorem *m-braids form a group Br_m given by generators $\sigma_1, \ldots, \sigma_{m-1}$ and relations:*

1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$,

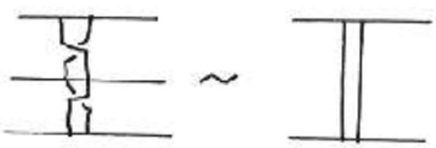
2) $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$.



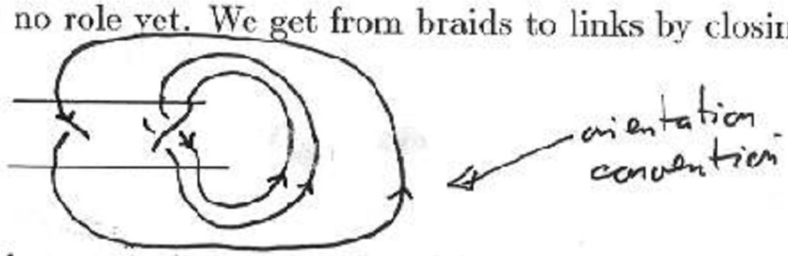


□

Remark Relation 2 corresponds to an R_3 move. $\sigma_i \sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$ corresponds to an R_2 move.

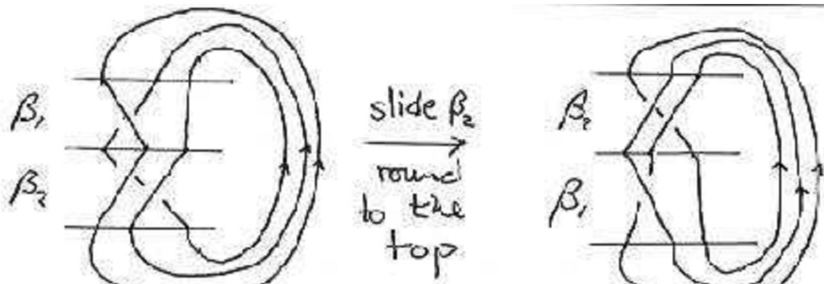


R_1 moves play no role yet. We get from braids to links by closing up:

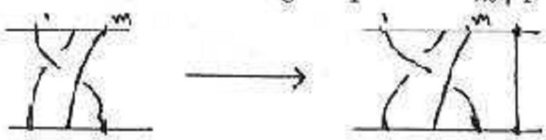


We will show that every link has a closed braid representation. Notice that if $\beta_1, \beta_2 \in Br_m$ then both $\beta_1\beta_2$ and $\beta_2\beta_1$ give the same link.

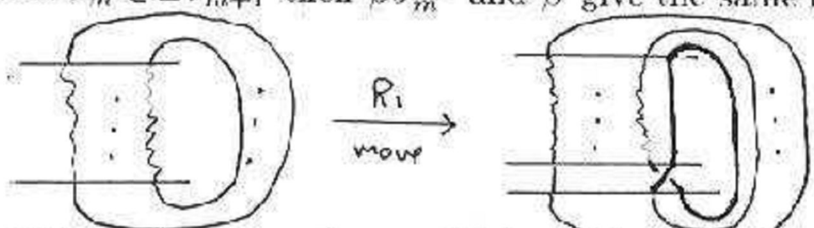
Previous lecture Next lecture An application of braids to Public Key Cryptosystems An application of braids to Quantum computers. Another reference. See also the April 2006 edition of Scientific American, *Computing with Quantum Knots* An old version of lecture 14 with proof of 6.6 in a special case



We can regard Br_m as a subgroup of Br_{m+1} by:



If $\beta \in Br_m$ and $\sigma_m \in Br_{m+1}$ then $\beta\sigma_m^{\pm 1}$ and β give the same link:



6.8 Theorem *(Markov) Isotopy classes of links are in one-one correspondence with equivalence classes of elements of the union of the set of braid groups:*

$$B = B_{r_1} \cup B_{r_2} \dots$$

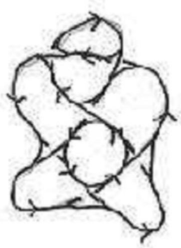
disjoint

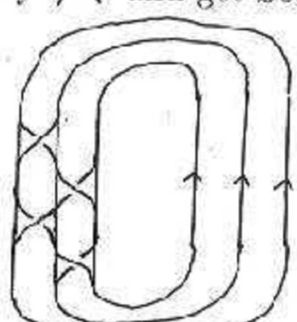
where the equivalence relation is generated by the two Markov moves:
M1) If $\beta_1, \beta_2 \in B_{r_m}$ *then then* $\beta_1 \ \beta_2 \sim \beta_2 \ \beta_1$
M2) If $\beta \in B_{r_m}$ *and* $\sigma_m \in B_{r_{m+1}}$ *then* $\beta \sim \beta\sigma_m^{\pm 1}$. □

6.9 Remark We have shown how to go from braid to link, next we show how to reverse the process. Knot theory is thus reduced to group theory. This does not solve the problem though.

Seifert circles
In an oriented diagram replace  and  by  and get *Seifert circles*:

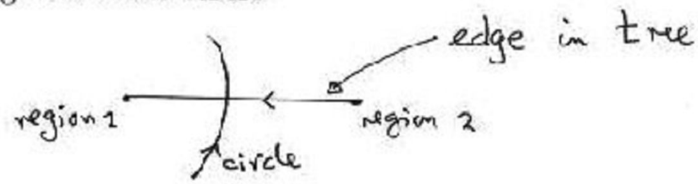
6.10 Example

6_2 

4_1 

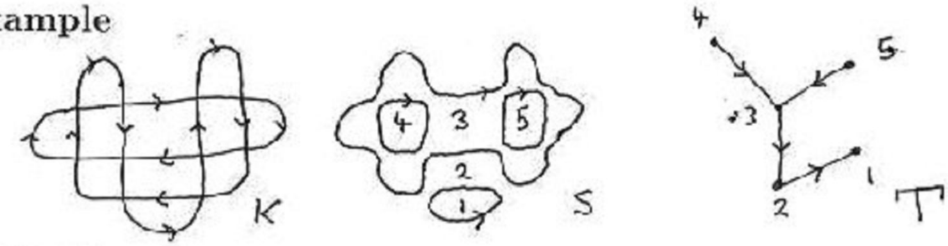
Define the *Seifert tree* associated with the Seifert circles by:
• A vertex for each region

• A directed edge for each circle
So that:



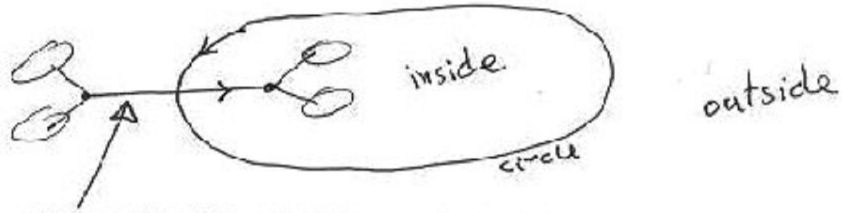
this is analogous to the quadrilateral construction for diagrams after 2.12.

6.11 Example



6.12 Remark

a) T is (always) a *tree* (no closed circuits) since cutting an edge leaves T in two pieces:

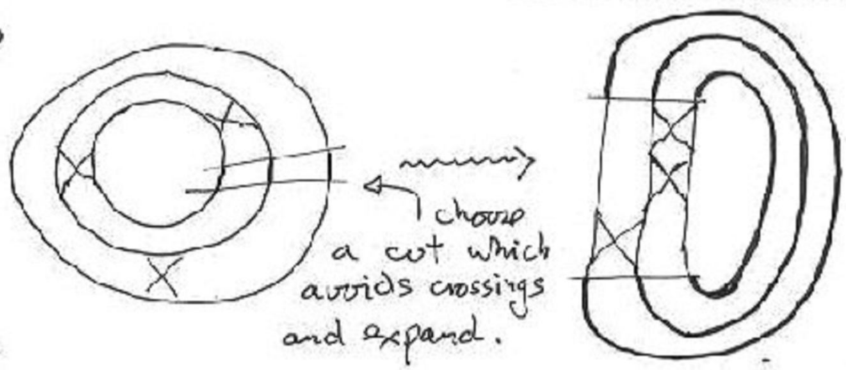


edge cannot be part of a closed circuit

b) The directions of Seifert circles agree at crossings and the circles are unchanged by crossing changes.

6.13 Lemma *An oriented diagram is a closed braid diagram (up to an R_0 move) if and only if its Seifert circles are concentric and all anticlockwise.*

Proof ($\Leftarrow$)



($\Longrightarrow$) ✓

□

Definition An R_∞ move (see examples question 10):

An outside arc has been lifted over the rest of the diagram. On a 2-sphere this would be an R0 move as the arc could be slid round the back through `infinity'. One could call this the `skipping rope' move.

A table of braid representations of knots

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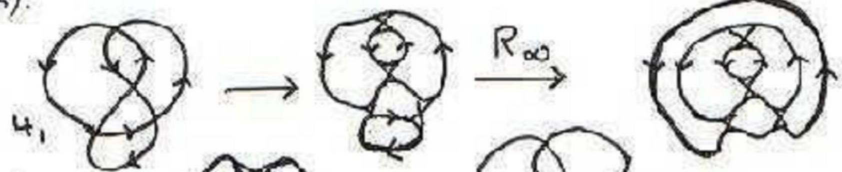
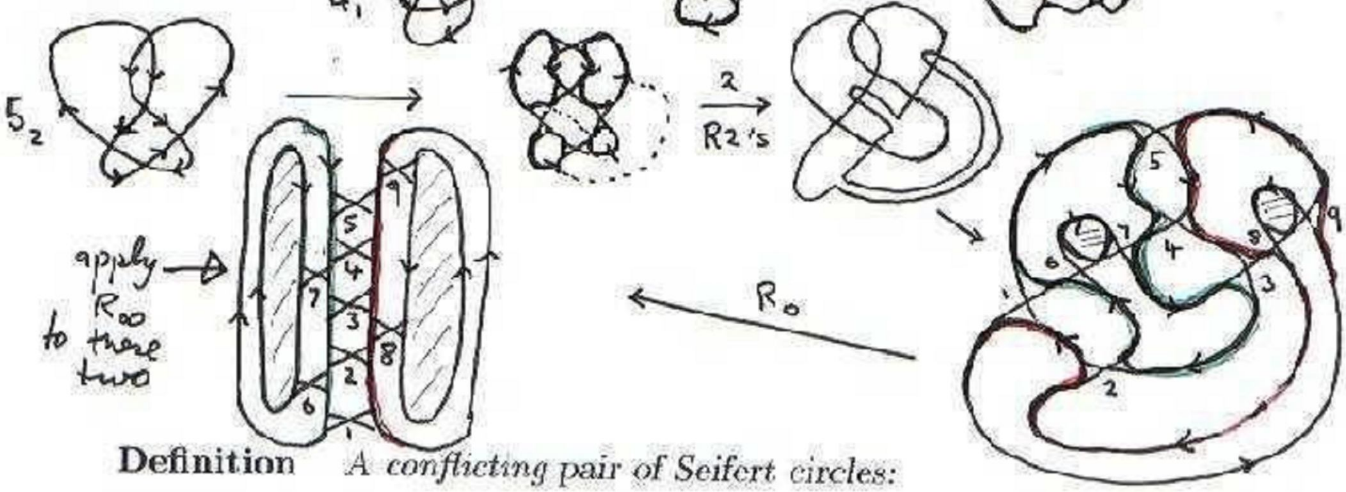
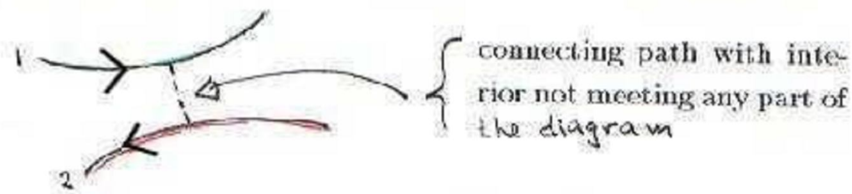
6.14 Theorem *Alexander (Vogel) Every oriented link has a closed braid representation. Furthermore the number of Seifert circles can be left unchanged and only R_3, R_2, and R_∞ moves are needed (on the corresponding diagrams).*



6.15 Examples



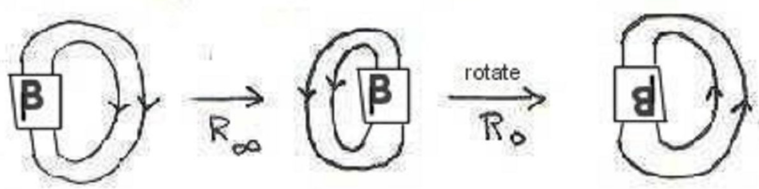
Definition *A conflicting pair of Seifert circles:*



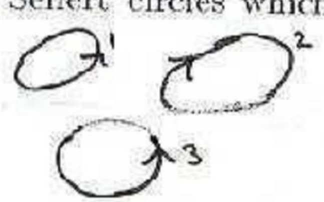
6.16 Lemma *An oriented link with no conflicting Seifert circles has a closed braid representation.*

Proof We claim that after R_0 and R_∞ moves we have the following possibilities:

Then we are done in case 1.
For case 2:



Proof of claim Consider a diagram of Seifert circles which has no conflicts. Find the outermost circles:



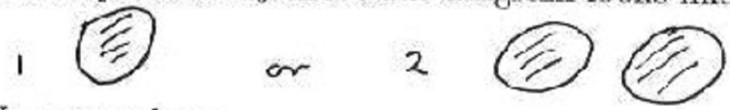
From knot to braid as in 6.15 but neater.

8_17 with Seifert circles

Here is a simple but inefficient proof.

Another proof of Alexander's theorem.

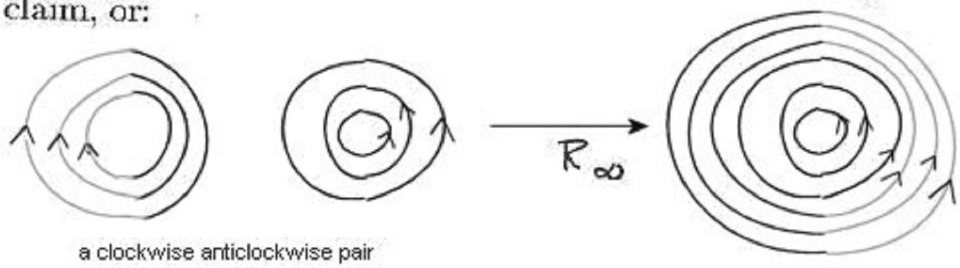
Three or more imply a conflict since if 1 and 2 agree then 3 must conflict with one of them. Hence up to an R_3 move our diagram looks like:



Look inside 1 say. We cannot have

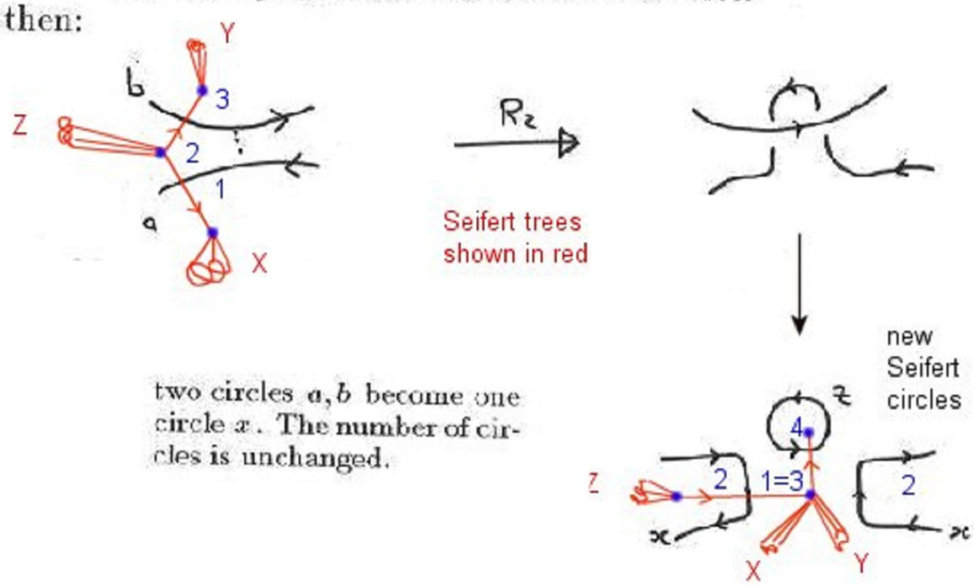
since some pair from $\{1, 3, 4\}$ must conflict. So we have:

Continuing in this way we are led to one of the two possibilities in the claim, or:

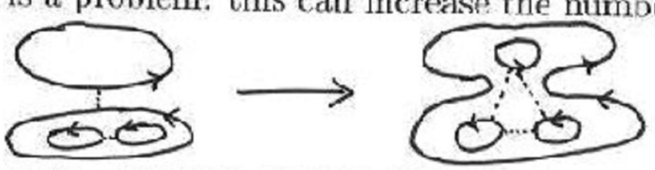


□

To prove the theorem we are reduced to removing all conflicts by R_2 moves which do not change the number of Seifert circles.
Consider then:



There is a problem: this can increase the number of conflicts:



However we will show that the following program terminates.

1. Find Seifert circles
2. If a conflict exists do an R_2 else END

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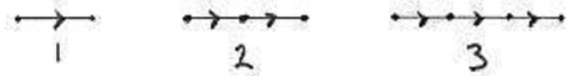
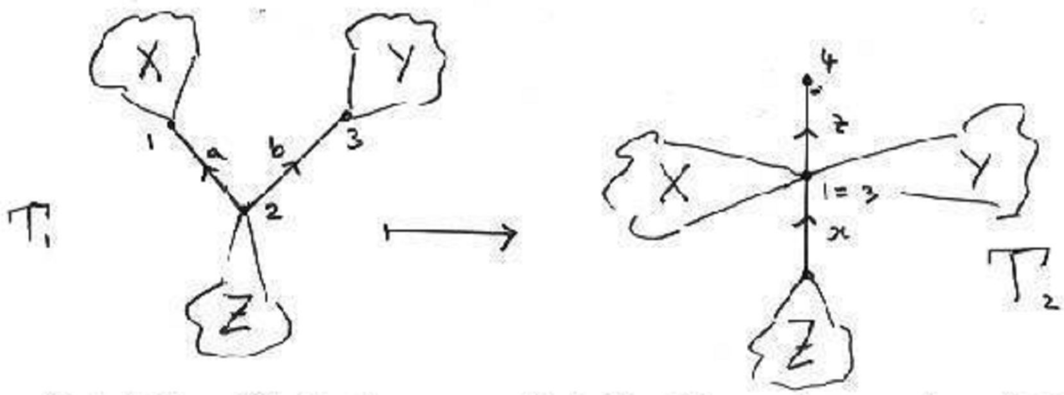
Definition A *chain* in an oriented tree is a collection of edges which together make up an (oriented) subdivision of an oriented interval as shown:

n-chains:



Consider the effect of the R_2 move on the Seifert tree:



Let $\phi\colon \mathrm{T}_1 \to \mathrm{T}_2$ be the map which identifies a, b to x. Let C_i be the set of chains in T_i, $i = 1, 2$. Define $\psi\colon C_1 \to C_2$ by

$$\psi(c) = \begin{cases} \phi(c) \cup z & \text{if } c \text{ terminates at } 3, \\ \phi(c) & \text{otherwise.} \end{cases}$$

6.17 Lemma *ψ is injective, ψ is not surjective.*

Proof Not surjective since the 1-chain z is not in the image. ψ is injective since the only way ϕ can identify distinct chains c_1, c_2 is if one terminates at 1 and the other terminates at 3 but then $\psi c_1 \neq \psi c_2$. □

The effect of the R_2 move then is to increase the number of chains in Γ without increasing number number of edges in T—hence eventually we must find no more moves are possible as required.
In case a,b,x,z come out with directions opposite to those shown change `terminates' to `starts' in the definition of ψ. □

Previous lecture Next lecture See the proof working on 8_{15} See also solutions 7

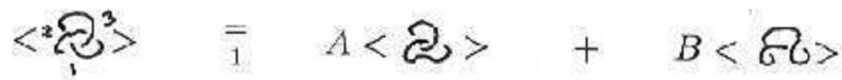
7 The Jones Polynomial

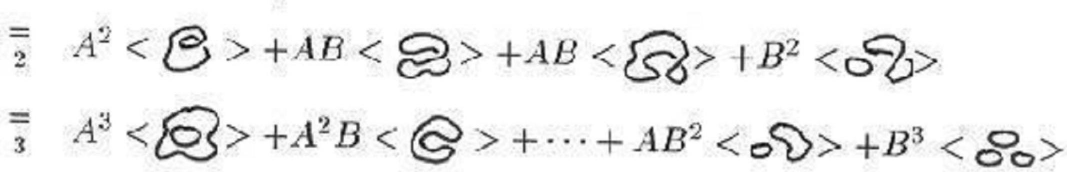
Given variables A, B attempt to 'define' the *bracket polynomial* of an unoriented diagram by:

$$\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle \qquad 7.1$$

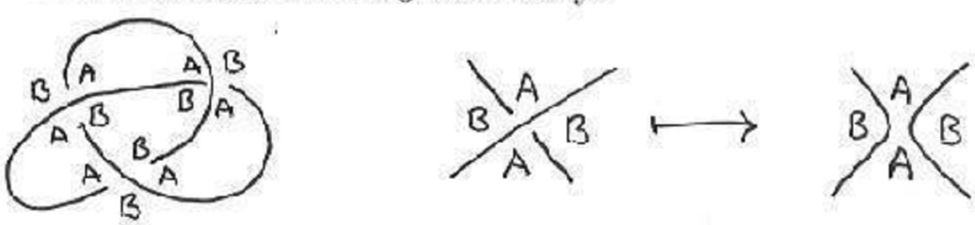
Only the part of the diagram which is changed is shown. Get $\begin{cases} A \\ B \end{cases}$ by turning $\begin{cases} \text{right} \\ \text{left} \end{cases}$ on the underpasses.

7.2 Example





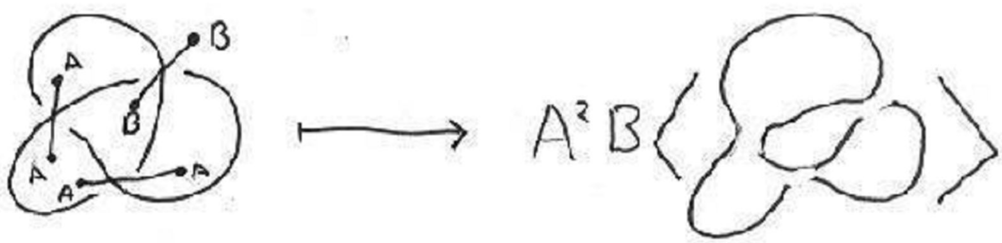
To aid calculation mark regions locally:



Multiply by A when A regions joined up.

A term in the final sum is determined by a choice of *splitting markers*:

e.g.



Terms in the polynomial can be read off from states of the diagram D.

A *state* S is a choice of splitting markers,one for each crossing:

The *coefficient* $< D|S >$ of state S is $A^i B^j$ where $\begin{cases} i \\ j \end{cases}$ is the number of markers joining the $\begin{cases} A \\ B \end{cases}$ regions.

Now suppose:

$$< O > = 1, \text{ where O is the standard unknot diagram}$$
$$< O \cup D > = d < D >, \quad D \text{ not empty,}$$

and let $|S|$ denote the number of circles in a state after splitting then

$$< D > \;=\; \sum_{\text{states } S} < D|S > d^{|S|-1} \qquad 7.3$$

This is a polynomial in variables A, B and d.

7.4 Examples

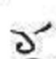
$<$  $>$ $= A <$  $> + B <$  $>$

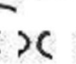
$= Ad + B$

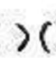
$<$  $>$ $= A^3 d + \cdots + B^3 d^2$

7.5 Proposition *$< - >$ is a regular isotopy invariant provided $B = A^{-1}$, $d = -A^2 - A^{-2}$.*

Proof We have to satisfy R_2 and R_3. For R_2

$<$  $> = A <$  $> + B <$  $>$

$= A(A <$  $> + B <$  $>) + B(A <$  $> + B <$  $>)$

$= (ABd + A^2 + B^2) <$  $> + AB <$  $>$

So we want $ABd + A^2 + B^2 = 0$, $AB = 1$,
and this follows from the hypothesis.

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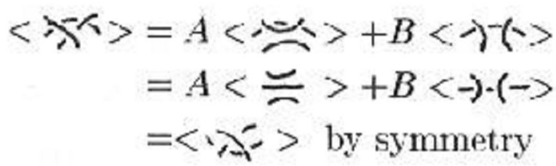
Now for R_3.

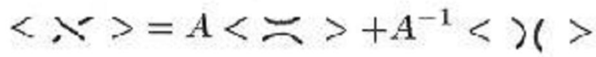


So R_3 follows from R_2! □

7.6 Conclusion. By putting $B = A^{-1}$ and $d = -A^2 - A^{-2}$, $< - >$ is a well defined (regular isotopy invariant) Laurent polynomial in A determined by the three equations:

$$< O > = 1$$

$$< O \cup D > = d < D >$$

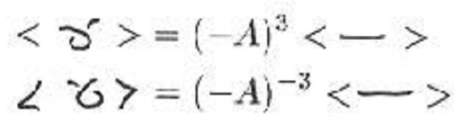


and given by $< D > = \sum_{\text{states } S} < D|S > d^{|S|-1}$.

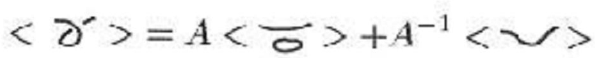
From now on assume $B = A^{-1}$ and $d = -A^2 - A^{-2}$. What about the R_1 move?

7.7 Proposition

i)
ii)



Proof of i)



$$= (dA + A^{-1}) < — >$$
$$= (-A^3 - A^{-1} + A^{-1}) < — >$$
$$= (-A)^3 < — >$$

□

Definition For an oriented link K define

$$X_K(A) = (-A)^{-3\omega(D)} < D >$$

where D is an (oriented)diagram for K.

7.8 Theorem *X_K is an isotopy invariant.*

Proof Since $\omega(D)$ and $< D >$ are unchanged by R_2 and R_3 moves it suffices to check R_1, but

$$(-A)^{-3\omega(\;)} < \; > = (-A)^{-3(\omega(\;)+1)}(-A^3) < \; >$$
$$= (-A)^{-3\omega(\;)} < \; > .$$

Similarly for . □

7.9 Proposition

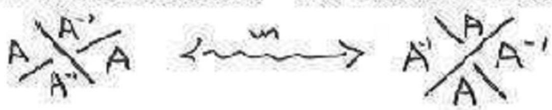
i) $< D > (A) =< mD > (A^{-1})$, mirror in plane of diagram

ii) $X_K(A) = X_{mK}(A^{-1})$,

iii) $X_K(A) = X_{rK}(A)$.

Proof

i) Immediate from definitions—the roles of A and A^{-1} are interchanged.



ii)

Now the writhe also changes sign, $\omega(D) = -\omega(mD)$. So $(-A)^{-3\omega(D)} = (-A^{-1})^{-3\omega(mD)}$.

iii) Writhe and labelling are unchanged:

□

7.10 Examples

i)

$$< D >=< \; > = A < \; > + A^{-1} < \; >$$
$$= A(-A)^3 < \bigcirc > + A^{-1}(-A)^{-3} < \bigcirc > \text{ by } 7.6$$
$$= -A^4 - A^{-4}$$
$$=< mD >$$

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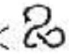
ii)

$< R3_1 > = <$  $> = A <$  $> + A^{-1} <$  $>$

$$= A(-A^4 - A^{-4}) < O > + A^{-1}(-A^{-3})^2 < O >$$

$$= -A^5 - A^{-3} + A^{-7}$$

therefore $\quad X_{R3_1} = (-A)^{-3.3} < R3_1 >$

$$= A^{-4} + A^{-12} - A^{-16}$$

so by 7.9 $\quad X_{L3_1} = A^4 + A^{12} - A^{16}$

From 7.8 then we have:

7.11 Proposition *The left and right trefoils $L3_1$ and $R3_1$ are knotted, are not isotopic, and have homeomorphic complements.*

Proof For the last part observe that reflection in a mirror provides a homeomorphism of any subspace of $\mathbb{R}^3$ with its image. □

Define the *Jones polynomial* for oriented links by $V_K(t) = X_K(t^{-\frac{1}{4}})$.

7.12 Proposition

i) $V_K(t) = V_{mK}(t^{-1})$,

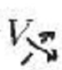
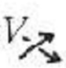
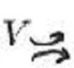
ii) $V_K(t) = V_{rK}(t)$.

Proof Immediate from 7.9 □

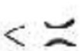
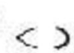
Conjecture V detects knots ie K knotted implies $V_K \neq 1$. This is true for knots with ≤ 17 crossings.

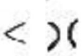
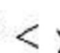
7.13 Theorem *(skein relations)*

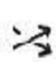
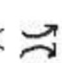
i) $V_O(t) = 1$

ii) $t^{-1}V$  $- tV$  $= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$ 

Proof ii)

$<$  $> = A <$  $> + A^{-1} <$  $>$

$<$  $> = A <$  $> + A^{-1} <$  $>$

therefore $\quad A <$  $> - A^{-1} <$  $> = (A^2 - A^{-2}) <$  $>$ 1

(orientation irrelevant so far but see what follows)

Recall $X_K(A) = (-A)^{-3\omega(D)} < D >$ so $< D > = (-A)^{3\omega(D)} X_K(A)$ 2

and

$$\omega(\times) = 1 + \omega(\asymp)$$
$$\omega(\times) = -1 + \omega(\asymp)$$ 3

Put 2 and 3 into 1:

$$A(-A)^{3(1+\omega(\asymp))} X_{\times} - A^{-1}(-A)^{3(-1+\omega(\asymp))} X_{\times}$$
$$= (A^2 - A^{-2})(-A)^{3\omega(\asymp)} X$$

Remove the common factor $(-A)^{3\omega(\asymp)}$:

$$-A^4 X_{\times} + A^{-4} X_{\times} = (A^2 - A^{-2}) X_{\asymp}$$

so
$$A^4 X_{\times} - A^{-4} X_{\times} = (-A^2 + A^{-2}) X_{\asymp}$$

Now put $A = t^{-\frac{1}{4}}$. □

7.14 Proposition *For links K and L,*

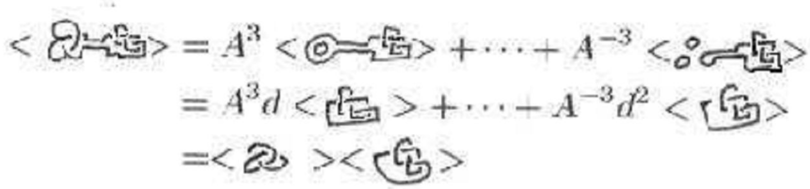
i) $V_{K\sharp L} = V_K . V_L$,

ii) $V_{K\cup L} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) V_K . V_L$.

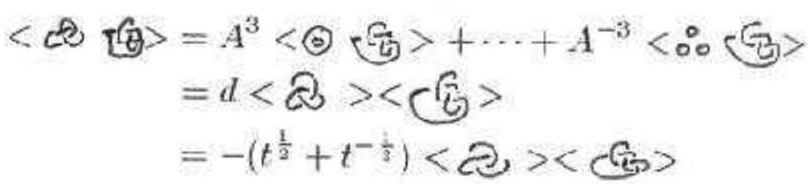
Proof In calculating V first operate on the crossings of K, and recall $< O \cup K > = d < K >$, $< O > = 1$.

eg (see 7.2) :

i)

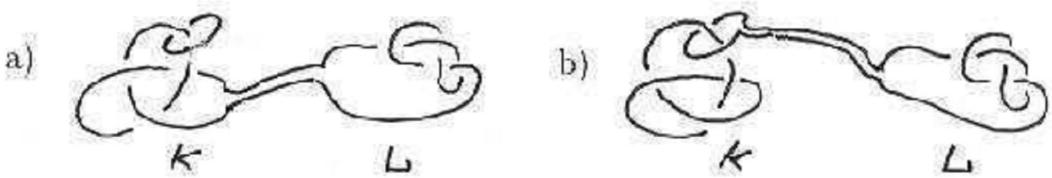


ii)



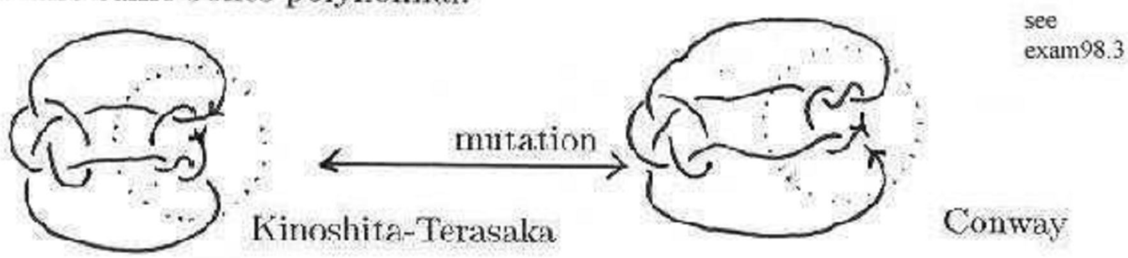
Since $\omega(K \cup L) = \omega(K) + \omega(\;\; L)$ the result follows from the definition.
$= \omega(K \# L)$
□

7.15 Example $K \# L$ is ambiguous for links. We can use this fact to find non isotopic links with the same Jones polynomial:



By 7.14 i) these two have the same Jones polynomial. They are not isotopic since a) has an unknotted component and b) does not.

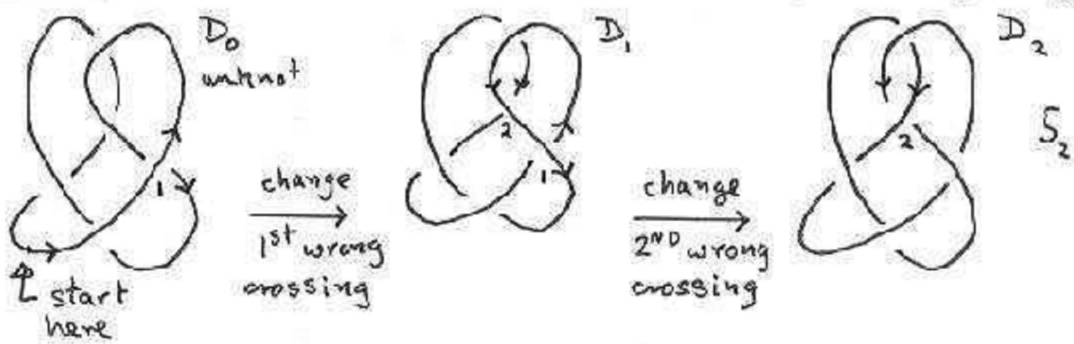
Similarly, using another kind of 'sum' one can show the following knots have the same Jones polynomial.



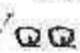
Also $V_{5_1} = V_{10_{132}}$ (Rolfsen notation)

Calculating V_K

We can use the skein relations and isotopy invariance to calculate V_K. For example suppose V_K calculated for all links with ≤ 4 crossings say, and we want to compute V_{5_2}. Consider a sequence of knots starting with the unknot and ending with V_{5_2}, each differing from its neighbour by a single crossing change:



$$t^{-1}V_{D_1} - tV_{D_0} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$$ 

$$t^{-1}V_{D_1} - tV_{D_0} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$$  by isotopy invariance

1) $$t^{-1}V_{D_1} - t = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})(-t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

2) $$t^{-1}V_{D_2} - tV_{D_1} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$$  only 4 crossings so known

Combine 1) and 2) to get:

$$V_{5_2} = t - t^2 + 2t^3 - t^4 + t^5 - t^6$$

A DOS Jones polynomial calculator, and a Java Jones polynomial calculator. For Kinoshita-Terasaka and Conway see exam 98 solution to 3 iv). Links which the Jones polynomial cannot distinguish from the trivial link do exist. See Thistlethwaite. A popular article can be found here: *Jones, Vaughan F. R. KNOT*

THEORY AND STATISTICAL MECHANICS; Scientific American, November 1990, page 52.

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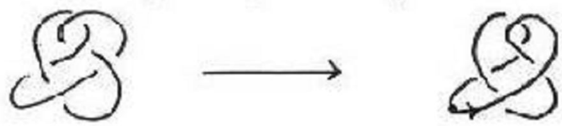
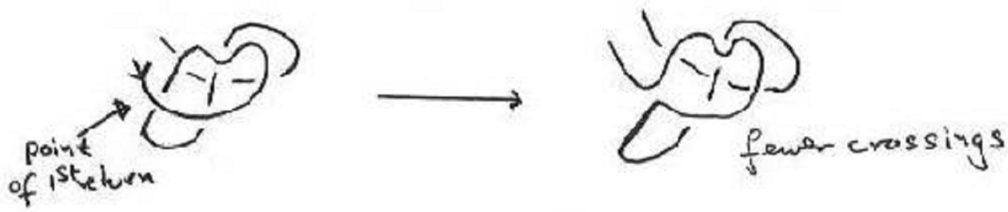
For the calculation in general we need the following lemma.

7.16 Lemma *Any knot diagram is a diagram for the unknot after suitable crossing changes.*

Proof Choose a starting point on an arc. Trace around the diagram going over at crossings not previously visited otherwise going under:



An induction on the number of crossings shows the result is the unknot (as in the proof of 6. 2).



□

The calculation algorithm

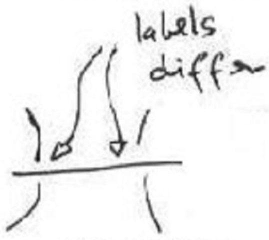
Let D be a diagram, with n crossings, for a knot K. Assume V_L calculated for any L which has a diagram with $< n$ crossings. Let D_0 be the diagram of the unknot produced by the lemma. Then there is a sequence of diagrams $D_0, \ldots, D_k$ all having the same shadow, where D_k is D and D_i for $0 \leq i \leq k$ is D_0 with the first i wrong crossings put right—a crossing is wrong if it is different from the corresponding crossing in D. Let K_i be the knot corresponding to D_i. Given $V_{K_{i-1}}$ then we can calculate V_{K_i} from the skein relation. The calculation of V_K thus rests on a double induction on i and n.

In describing the algorithm we assumed K was a knot.
A similar treatment works for links. □

8 Alternating Links

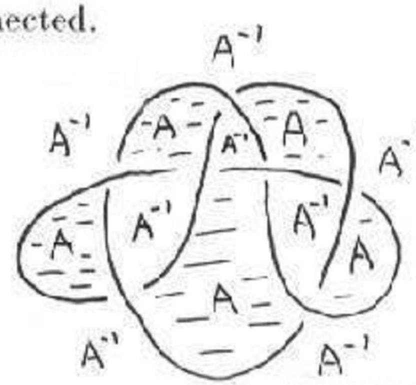
Assume diagrams are reduced alternating and connected.

Key observation:



not alternating

alternating



alternating

Entire region gets a consistent label in the alternating case

(the rule is still: turn right on undercrossing to join A regions)

We can assume a chess boarding chosen so black regions are marked A and white regions are marked A^{-1}. Let $V =$ number of crossings, $W =$ number of White regions, $B =$ number of Black regions.

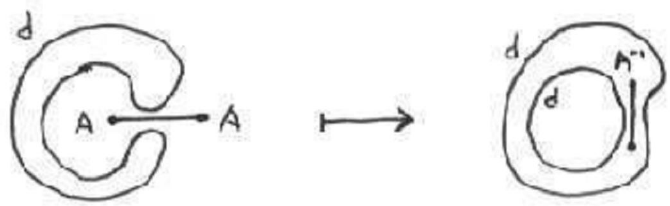
8.1 Proposition *Let D be alternating reduced and connected. Then the term in $< D >$ of* $\begin{cases} \text{highest} \\ \text{lowest} \end{cases}$ *degree in A is* $\begin{cases} (-1)^{W-1}A^{V+2W-2} \\ (-1)^{B-1}A^{-V-2B+2} \end{cases}$

Proof Let S be the state obtained by splitting every crossing in the A direction, then $< D|S >= A^V$ and $|S| = W$. So S contributes the term (see 7.3) $A^V d^{|S|-1} = A^V d^{W-1}$.

Since $d = -A^2 - A^{-2}$ the highest degree term here is $(-1)^{W-1}A^{V+2W-2}$. Consider another state S'. We get S' From S by switching a subset of the markers of S. So there is a sequence $S = S(0), S(1), \ldots, S(n) = S'$ where $S(i+1)$ is obtained from $S(i)$ by switching one marker of type A to type A^{-1}.

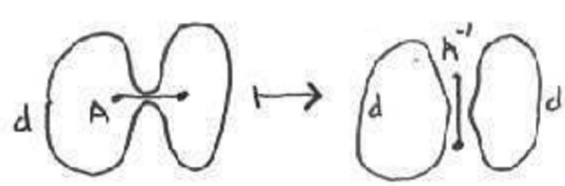
Either

i) 1 circle splits:



Contribution before

$Ad = A(-A^2 - A^{-2}) = -A^3 - A^{-1}$



Contribution after

$A^{-1}d^2 = A^{-1}(-A^2 - A^{-2})^2 = A^3 +$ lower terms

The proof of Proposition 8.1 illustrated. Celtic knots, and a quote from Tait.

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The highest degree term is unchanged (apart from sign).

Or

ii) 2 circles join:

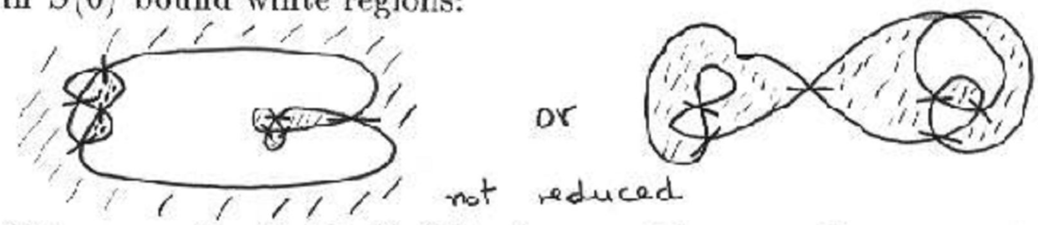
The contribution before is

$$Ad^2 = A(-A^2 - A^{-2})^2 = A^5 + \text{ lower terms}$$

and the contribution after is

$$A^{-1}d = A^{-1}(-A^2 - A^{-2}) = -A - A^{-3}.$$

The highest degree term now has lower degree. So in either case the maximal degree term either stays the same or decreases. But it must decrease at the first step $S(0) \mapsto S(1)$ since D is reduced and all circles in $S(0)$ bound white regions:



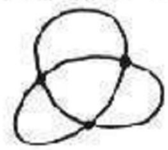
This proves the first half of the theorem (the max. degree term in $< D >=$ max. degree term in $S(0)$). The second half is proved similarly.
□

8.2 Example Find the highest and lowest power of t occurring in $V_{5_2}(t)$.

In $X_{5_2}(A) = (-A)^{-3\omega(D)} < D >$ the highest power of A has exponent $-3\omega + V + 2W - 2 = -15 + 5 + 8 - 2 = -4$. Therefore the lowest power of t in $V(t) = X(t^{-\frac{1}{4}})$ has exponent 1.

In $X_{5_2}(A)$ the lowest power of A has exponent $-3\omega - V - 2B + 2 = -15 - 5 - 6 + 2 = -24$. Therefore the highest power of t in $V(t)$ has exponent 6. This agrees with our earlier calculation.

8.3 Lemma *For an alternating connected diagram D we have $R = V + 2$, where $R =$ number of regions.*



Proof Consider the shadow of D as a graph of vertices and edges. Then Euler's Theorem gives $V - E + R = 2$ Now $V =$ number of crossings, number of arcs $= E/2$ since D is alternating. We also have number of arcs $=$ number of crossings so $V - 2V + R = R - V = 2$.

□

Definition The *span* of a Laurent polynomial is *max degree* − *min degree.*

Then

$$\begin{aligned} span < D > &= (V + 2W - 2) - (-V - 2B + 2) \\ &= 2V + 2(W + B) - 4 \\ &= 2V + 2R - 4 \\ &= 2V + 2(V + 2) - 4 \\ &= 4V \end{aligned}$$

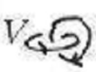
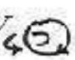
Hence from the definition of the Jones polynomial we have;

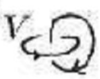
8.4 Theorem *If a link L has a connected reduced alternating diagram D then $Span(V_L(t)) = V$, where V is the number of vertices in D.* □

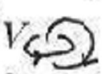
8.5 Example Right Hopf link (linking number = 1)

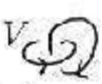
$V = 2$, $W = 2$, $B = 2$, $\omega = 2$. max term in $X_{Hopf}(A)$ is $(-A)^{-3\omega}(-1)^{W-1}A^{V+2W-2} = -A^{-2}$ so the min term in $V_{Hopf}(t)$ is $-(t^{-\frac{1}{4}})^{-2} = -t^{\frac{1}{2}}$.

Check:

$t^{-1}V$  $- tV$  $= (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V$ 

$t^{-1}V$  $- t(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

$t^{-1}V$  $= -t^{\frac{3}{2}} - t^{\frac{1}{2}} + t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

V  $= -t^{\frac{5}{2}} - t^{\frac{1}{2}}$

This confirms that the min deg term is $-t^{\frac{1}{2}}$ and the span is $\frac{5}{2} - \frac{1}{2} = 2$ which is the number of crossings in the diagram as predicted by 8.4. Further by 7.12 we see that V distinguishes right and left Hopf links.

A short list of Alexander and Jones polynomials. Jones polynomials of knots up to 10 crossings together with their Dowker-Thistlethwaite codes.

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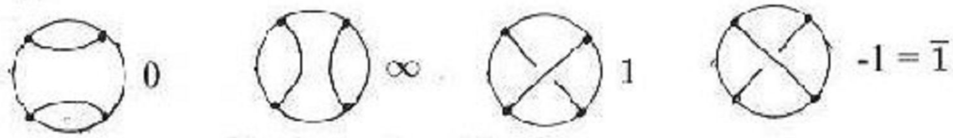
9 Tangles

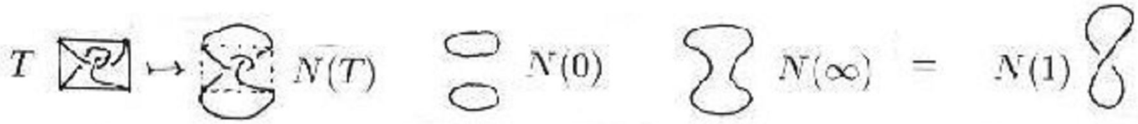
A *tangle* is a link in a ball with 4 fixed ends as shown. Isotopy of tangles (ends fixed) corresponds to equivalence of diagrams (in a disc) via R moves as usual.

Simple tangles:

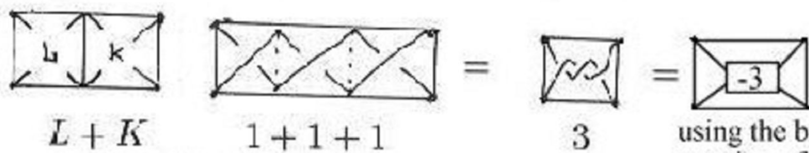


Or draw them like this:

If T is a tangle its *numerator* is the link $N(T)$ given by:



Tangles can be added:



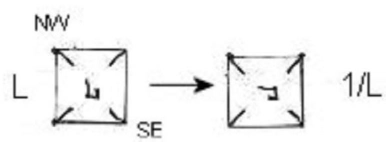
using the box notation of 6.6

Notice the left twists so eg $N(3) = Left3_1$.

There are two unary operations:

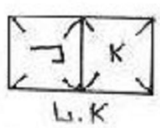
i) reflection in the plane of the diagram T ⟶ -T

ii) reflection in the NW SE diagonal T ⟶ 1/T



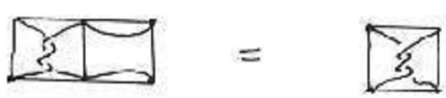
Tangles can be multiplied:

$T.K = \frac{1}{T} + K$



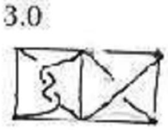
This makes little sense but is convenient notation

9.1 Examples

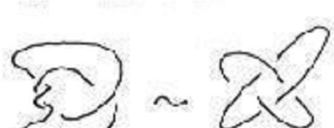


notice right twists

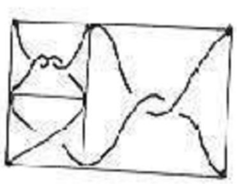
3.0



3.1



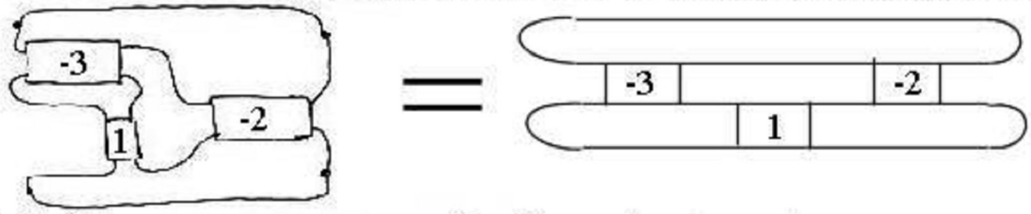
N(3.1)



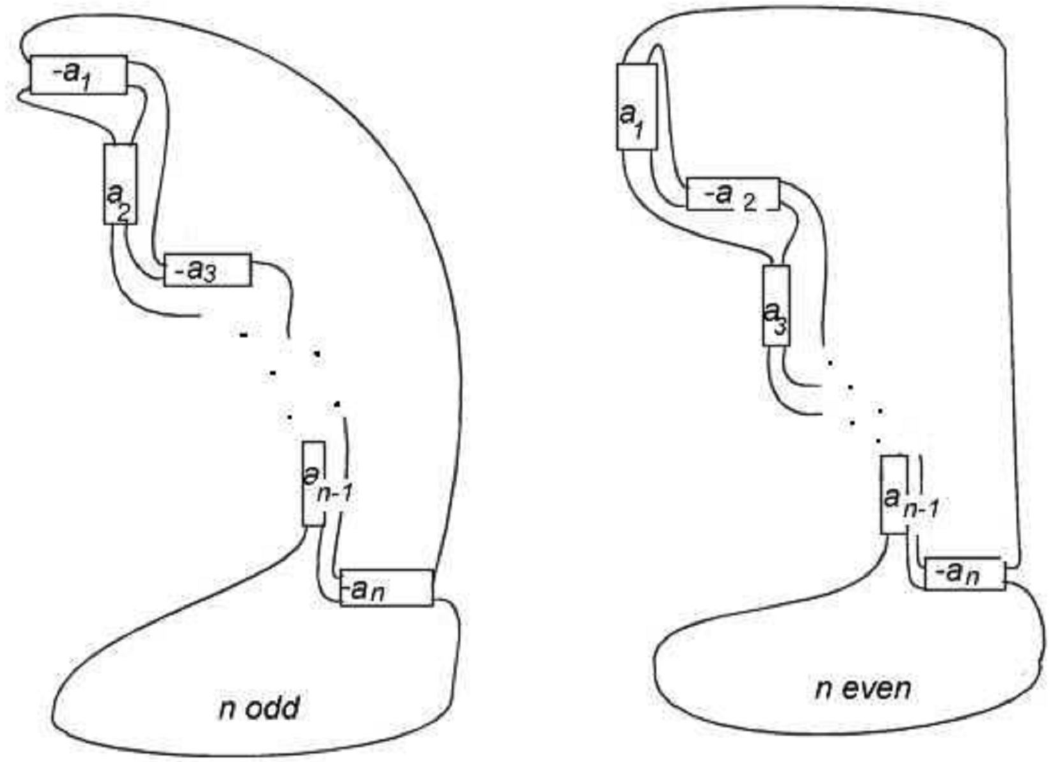
(3.1).2

$N(3\ 1\ 2) = 6_2$

Note that in the number in box notation of 6.6 $N((3.1).2)$ would be:



Definition $a_1 a_2 a_3 \cdots a_n = ((\cdots((a_1.a_2).a_3)\cdots a_n)$,where $a_i \in \mathbb{Z}$, is a *rational tangle*. With notation as in section 6 $N(a_1 a_2 a_3 \cdots a_n)$ is:

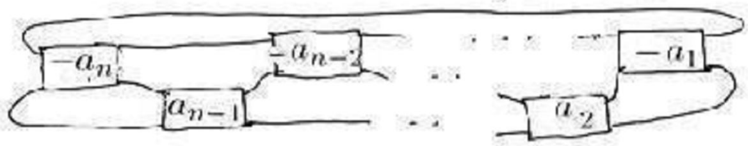


In either case we end with a horizontal box, and the diagram is alternating if all a_i have the same sign.

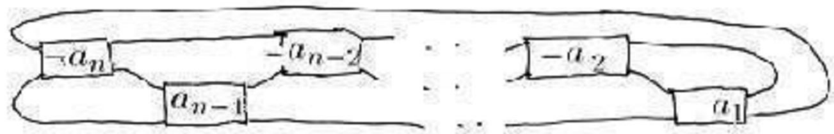
9.2 Proposition

$$N(a_1 a_2 a_3 \cdots a_n) = D(-a_n, a_{n-1}, -a_{n-2}, \ldots, (-1)^n a_1)$$

Proof **Recall after 6.6 we used flips and flypes to show:** $D(-a_n, a_{n-1}, -a_{n-2}, \ldots, (-1)^n a_1)$ is



n odd



n even

Turn these end over end **(a pancake flip)** to see the result. □

9.3 Corollary

$$N(a_1 a_2 a_3 \cdots a_n) = \begin{cases} N(a_n a_{n-1} \cdots a_1) & n \text{ odd} \\ N(-a_n \; -a_{n-1} \cdots -a_2 \; -a_1) & n \text{ even} \end{cases}$$

Refer to the definition before 6.6 (lecture 14)

Proof Turn $D(a_1, a_2, a_3, \cdots, a_n)$ end over end to see $D(a_n, a_{n-1}, a_{n-2}, \ldots, a_1)$
□

- [Knotation](#) A lecture by [John Conway](#).
- Pictures of [rational knots](#) 3_1 through 8_1 with their Conway notation.
- An aid to understanding [rational/2-bridge knot construction](#) and notation.
- Additional material on [rational tangles](#)
- Some history and pretty pictures of [rational knots](#)

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9.4 Proposition $\det N(a_1 a_2 a_3 \cdots a_n) = p$ where

$$\frac{p}{q} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\cdots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}}$$

$p \geqslant 0$ and p, q are coprime.

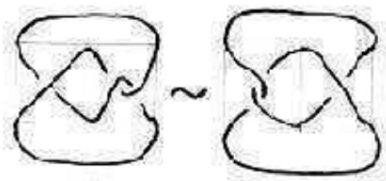
p/q is the *fraction* of the tangle $a_1 a_2 a_3 \cdots a_n$

□

9.5 Examples

i) $N(1\ 2) \sim N(-2\ \ -1) \sim L3_1$

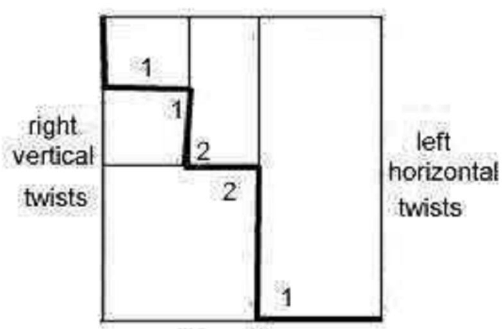
2 + 1/1 = 3/1 -1 +1/-2 = -3/2 det = 3



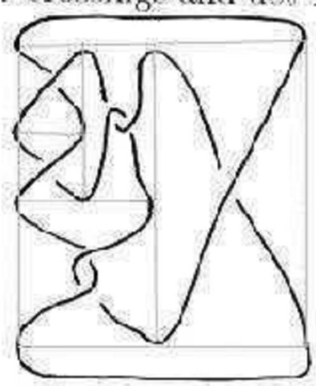
ii) $N(1\ 1\ 2\ 2\ 1) \sim N(1\ 2\ 2\ 1\ 1) \sim 7_5$ or $m7_5$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{17}{12} \qquad 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1}}}} = \frac{17}{10}$$

det = 17 and the only alternating knot with 7 crossings and det 17 is 7_5

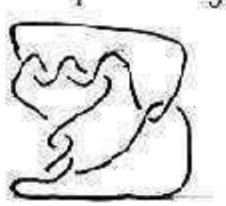


Template



or m 7_5

iii) $N(3\ 3\ 2) \sim 8_6$ $\frac{?}{q} = 2 + \frac{1}{3 + \frac{1}{3}} = 2 + \frac{3}{10} = \frac{23}{10}$, $\det 8_6 = 23$.



Here is an example of the calculation of a continued fraction corresponding to a rational number:

$$\frac{17}{10} = 1 + \frac{7}{10} = 1 + \frac{1}{\frac{10}{7}} = 1 + \frac{1}{1+\frac{3}{7}} = 1 + \frac{1}{1+\frac{1}{\frac{7}{3}}} = \boxed{1} + \cfrac{1}{\boxed{1} + \cfrac{1}{\boxed{2} + \cfrac{1}{\boxed{3}}}}$$

Notice the connection with the Euclidean algorithm:

17 = 10. 1 + 7
10 = 7. 1 + 3
7 = 3. 2 + 1
3 = 1. 3

9.6 Theorem *There is a bijection between (isotopy classes of) rational tangles and the rational numbers (including* $\infty = \frac{1}{0}$*) given by*

$$a_1, \ldots, a_n \mapsto a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\cdots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}}$$

□

9.7 Corollary *Any rational tangle has a standard form: either one of* $0, 1, -1, \infty$ *or* $a_1, \ldots, a_n$ *where,* $|a_1| \geq 2$*, all the* a_i *have the same sign except that* a_n *might be zero.*

Proof This follows from the standard form for continued fractions. If we write $[a_n, \ldots, a_1]$ for the continued fraction in the theorem then $[a_n, \ldots, a_2, 1] = [a_n, \ldots, a_3, a_2 + 1]$, $-1 = [-1]$, $0 = [0]$, $1 = [1]$, $\infty = [1, 0]$. To get the continued fraction $[a_n, \ldots, a_1]$ for $x \in \mathbb{Q}$, $x > 0$ write $x = a_n + b_n$, $0 \leq b_n < 1$, $\frac{1}{b_n} = a_{n-1} + b_{n-1}$ etc. □

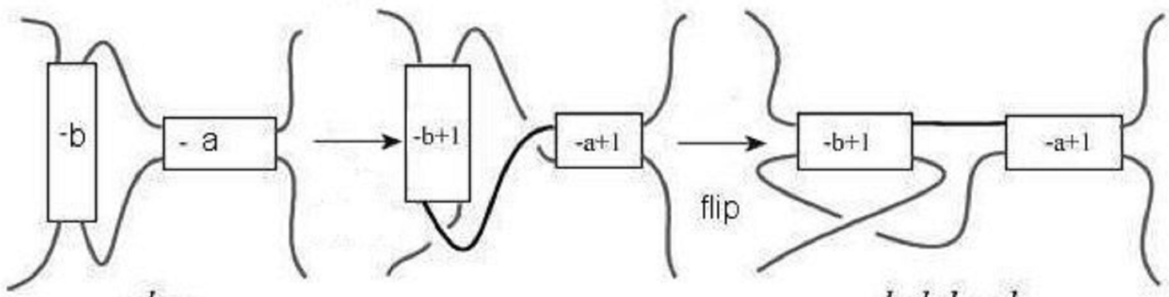
9.8 Example

i) In examples *9.5* the tangles correspond to different rational numbers and are thus not isotopic although their numerators are isotopic.

ii) The formula of Lagrange:

$$a + \frac{1}{-b} = (a-1) + \frac{1}{1 + \frac{1}{(b-1)}}$$

corresponds to an isotopy of tangles:



-b.a *b-1.1.a-1*

The formula can be used to convert to standard form. eg 232 and 2111 are isotopic --- note that the formula holds for any b, not just an integer b.

$$2 + \frac{1}{-3 + \frac{1}{2}} \qquad a = 2, b = 3 - \frac{1}{2}$$

$$= 1 + \frac{1}{1 + \frac{1}{2 - \frac{1}{2}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = \frac{8}{5}$$

($a = 2, b = 2$)

$2\,\bar{3}\,2$ $\bar{2}\,2\,1\,1$ $1\,1\,1\,1\,1 = 2\,1\,1\,1$

$N(11111)$ is the Whitehead link.

In general the flip can be used to convert eg *x.y.z.-b.a.u.v.w* to *-x.-y.-z.b-1.1.a-1.u.v.w* and prove injectivity in 9.6

surjectivity is obvious the difficult bit is proving well defined.

Example 9.9 A daisy chain D(a b c d) after 3 double flypes is the numerator of the tangle:
-(a+1).1.-(b+2).1.-(c+2).1.-(d+2).1.-(e+1) = a.-(b+1).1.-(c+2).1.-(d+2).1.-(e+1)
= -a.b.-(c+1).1.-(d+2).1.-(e+1)

= a.-b.-c.-(d+1).1.-(e+1)
= -a.b.-c.d.-e

An extract on Continued Fractions from David Fowler's book

Additional material including a proof of proposition 9.4

More on the use of the Lagrange identity and the proof of 9.6 can be found in a paper on rational tangles by Goldman and Kauffman. See pages 315 and 316. See also Kauffman and Lambropoulou.

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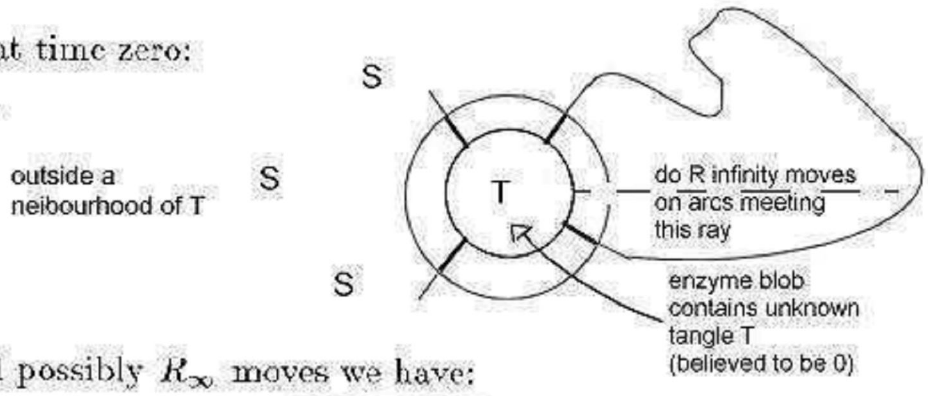
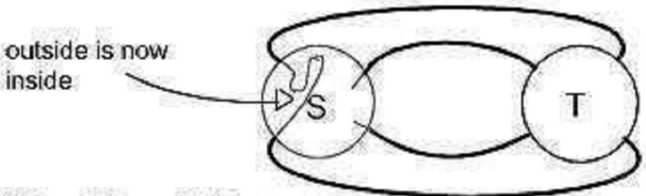
Enzyme action on DNA (sketch)

Manufacture lots of unknotted loops of DNA. Add a suitable enzyme. See what comes out as time passes. Observe first Hopf links, then figure eight knots, then Whitehead links, then some 6_2's. How does the enzyme do this?

Observation at time zero:



After R_0 and possibly R_∞ moves we have:



where $N(S+T) = N(1)$ is the unknot.

Assume the enzyme destroys T and replaces by R and then keeps on adding further R's. Also assume S T R are rational tangles. Then experiment shows:

i) $N(S+T) = N(1)$ unknot

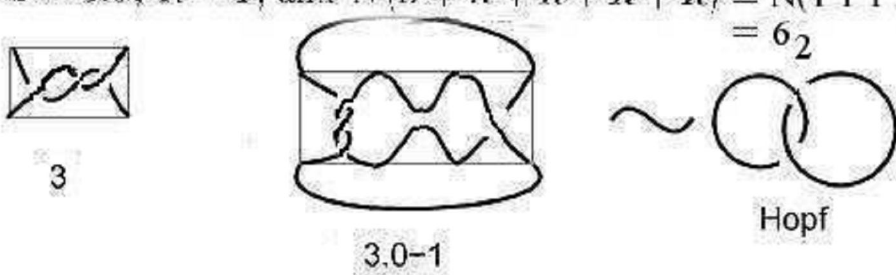
ii) $N(S+R) = N(2)$ Hopf

iii) $N(S+R+R) = N(2\ 1\ 1)$ 4_1

iv) $N(S+R+R+R) = N(1\ 1\ 1\ 1\ 1)$ the (+) Whitehead link

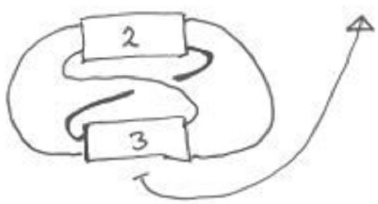
9.9 Theorem $S = 3.0$, $R = 1$, and $N(S+R+R+R+R) = N(1\ 1\ 1\ 2\ 1)$ $= 6_2$ □

Check ii)

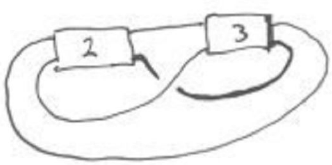


6_2

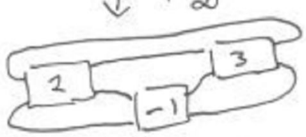
DNA Loop at 'time' 4



$\downarrow R_0$



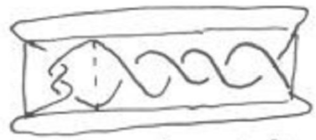
$\downarrow R_\infty$



$\sim mN(2.1.3)$

compare $N(S+R+R+R+R)$

$S = 3.0$ $R = -1$



$N(3.-4)$

$= m\,N(-3.4)$

$= m\,N(2.1.3)$

(recall $-p.q = (p-1).1.(q-1)$)

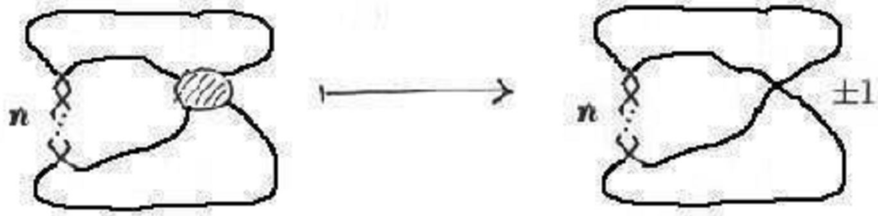
On the other hand:

$N(1\,1\,1\,2\,1) =$

$\overset{R_0}{\longmapsto}$ 6_2 above

OR $N(11121) \sim N(12111) \sim N(3111) \sim mN(1113) \sim mN(2.1.3)$

9.10 Remarks $S = 3.0$ and $R = -1$ satisfies ii) and iii). It is easy to see that this is the only solution if we add the hypothesis $S = \boldsymbol{n}.0$, $\boldsymbol{n} \in \mathbb{Z}$, and $R = \pm 1$. ie assume:



Then

$$\text{fraction}(S + R) = \frac{1}{\boldsymbol{n}} \pm 1,$$
$$\text{fraction}(S + R + R) = \frac{1}{\boldsymbol{n}} \pm 2.$$

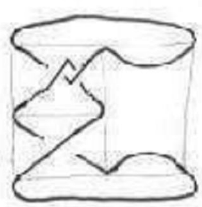
Now det(Hopf)=2 and det 4_1 =5, so we have by 9.4

$\frac{1}{n} \pm 1 = \frac{2}{p}$, $\frac{1}{n} \pm 2 = \frac{5}{q}$ for some none zero p and q with 2, p coprime and 5,q coprime. The only possibilities are

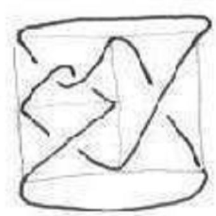
a) $\frac{1}{3} - 1 = \frac{2}{-3}$, $\frac{1}{3} - 2 = \frac{5}{-3}$ or b) $\frac{1}{-3} + 1 = \frac{2}{3}$, $\frac{1}{-3} + 2 = \frac{5}{3}$.

Check b) -3.1 = 2.1.0 -3.2 = 2.1.1 (recall -s.t = (s-1).1.(t-1))

$\frac{2}{3} = 0 + \frac{1}{1+\frac{1}{2}}$ $\frac{5}{3} = 1 + \frac{1}{1+\frac{1}{2}}$



Hopf



4_1

a) is equally valid - both Hopf and 4_1 are achiral.
But condition iv) shows the only possibility is a) as required ,since the Jones polynomial distinguishes the (+)-Whitehead link from the (-)-Whitehead link □

(See exam 2003 Q5)

- Whitehead links
- Knots and enzyme action on DNA strings (the source for the lectures - mirror all crossings to get agreement with the course).
- Pictures of Knotted DNA.
- Design any knot from DNA.

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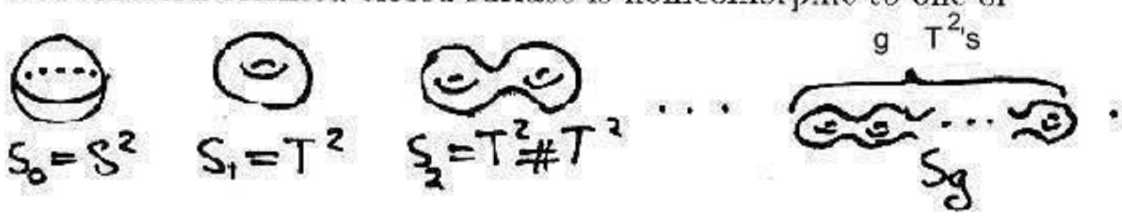
10 Genus and knot sum

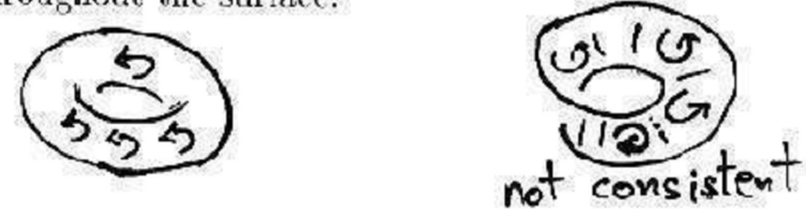
Oriented surface facts

A connected oriented closed surface is homeomorphic to one of

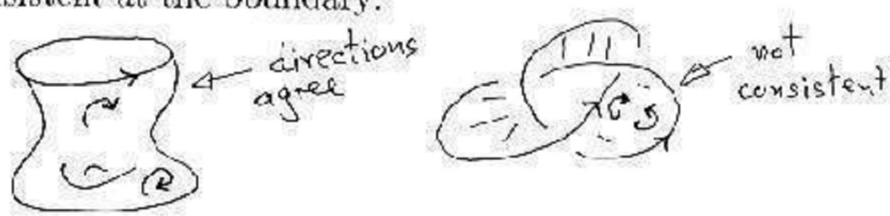


The *genus* of S_g is g. The Euler characteristic of S_g is calculated from the formula $\chi = V - E + T$ after dividing the surface up into triangles, where V is the number of corners, E is the number of edges, and T is the number of triangles. In particular $\chi(\text{disc}) = 1$, $\chi(\text{Möbius band}) = 0$, **and** $\chi(S_g) = $ **2-2g**

An *orientation* of a surface is determined by a consistent clockwise/anticlockwise choice throughout the surface.



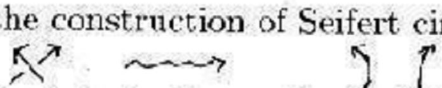
An oriented knot is the boundary of an oriented surface in $\mathbb{R}^3$ if the surface is homeomorphic to an S_g with disc removed, and orientations are consistent at the boundary:

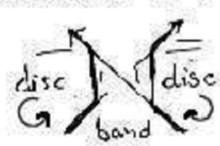


Warning: the trefoil bounds a triple twisted band—not orientable.

10.1 Proposition *Any oriented knot bounds an oriented surface in* $\mathbb{R}^3$.

Proof Recall the construction of Seifert circles 6.10

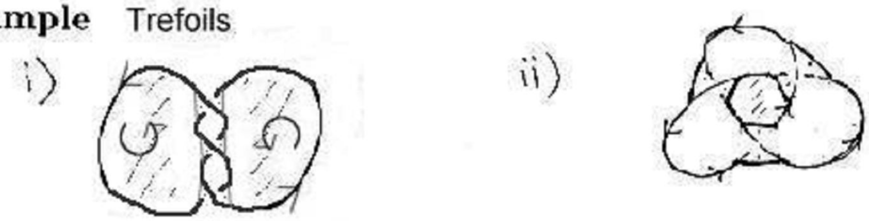




Bound each Seifert circle by a disc and join the discs with bands to get an oriented Seifert surface with knot as boundary.

If discs occur inside each other separate by lifting up to get 'nested igloos'. □

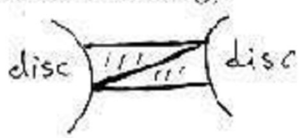
10.2 Example Trefoils



In case ii) put a dome on the outer circle. In either case get two discs joined by three bands. It follows from the next proposition that this is S_1 (with a hole).

10.3 Proposition *The surface constructed in 10.1 is homeomorphic to an S_g (with disc removed) where $g = \frac{1}{2}(c - s + 1)$ and $c = \sharp$ crossings, $s = \sharp$ Seifert circles.*

Proof We have $\chi(S_g -$ disc$) = (2 - 2g) - 1 = 1 - 2g$, $\chi($disc$) = 1$, $\chi($s discs$) = s$. Add a band at each crossing and get three new edges and two new triangles (at each crossing).



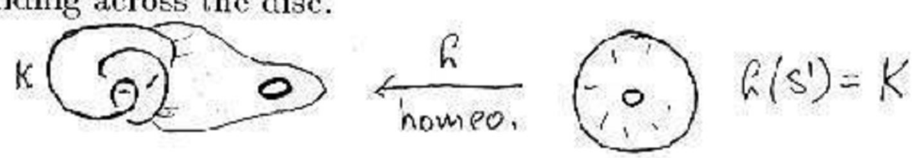
Therefore χ is increased by $0 - 3 + 2 = -1$ at each crossing. So $\chi($surface$) = s - c = 1 - 2g$. Hence the result. □

Definition The *genus* of a knot is the minimum genus of all connected orientable surfaces spanning the knot (genus surface with hole := genus surface without hole.)

10.4 Corollary *If D is a diagram for K with c crossings and s Seifert circles then $g(K) \leq \frac{1}{2}(c - s + 1)$* □

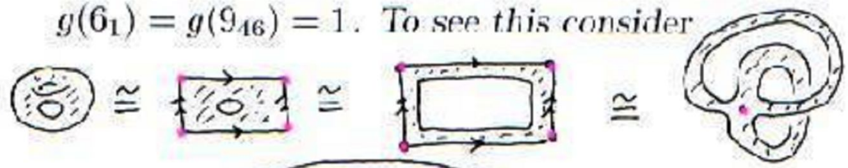
10.5 Proposition *$g(K) = 0$ If and only if K is the unknot.*

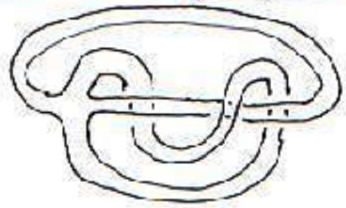
Proof Suppose $g(K) = 0$ then K bounds a disc, isotop K to a small unknot by sliding across the disc.



Conversely if K is the unknot K bounds a disc and $g(K) = 0$. □

10.6 Examples $g(6_1) = g(9_{46}) = 1$. To see this consider

$S_1 \setminus disc \cong$ 

Now recall 3.8 eg 9_{46} 

10.7 Theorem For knots K, L $g(K \sharp L) = g(K) + g(L)$.

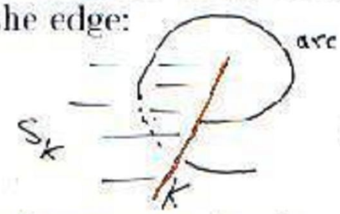
10.8 Corollory *If K is not the unknot then $K \sharp L$ is not the unknot for any L.*

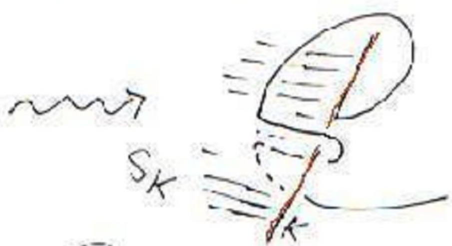
Proof Suppose $K \sharp L$ is the unknot then $0 = g(K \sharp L) = g(K) + g(L) \neq 0$ since $g(K) \neq 0$. □

Proof of theorem

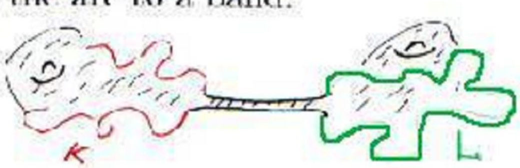
Step 1 $g(K \sharp L) \leq g(K) + g(L)$

Join a min genus S_K in $x < 0$ for K to a min genus S_L in $x > 0$ for L by an arc from K to L. If the arc meets S_L or S_K in its interior divert it over the edge:

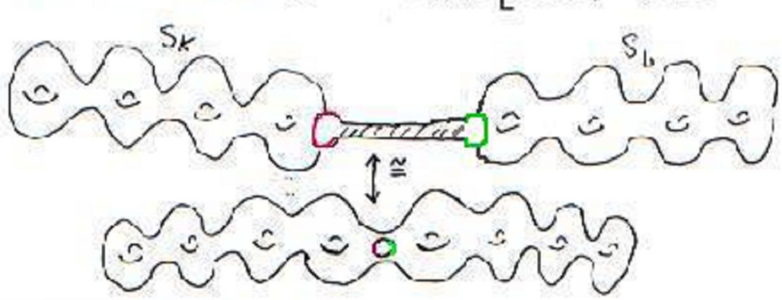




Thicken the arc to a band:



Then $K \sharp L$ is spanned by S with $g(S) = g(S_K) + g(S_L)$ since S is homeomorphic to $(S_K \cup \text{disc}) \sharp (S_L \cup \text{disc}) \setminus \text{disc}$



The trefoil knot bounds a torus with a hole. Here is a neat program which displays Siefert surfaces.

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11 The Conway polynomial

11.1 Theorem *For each oriented link K there is a polynomial $\nabla_K(z)$ satisfying:*

i) *K isotopic to L implies $\nabla_K = \nabla_L$,*

ii) *$\nabla_K = 1$ for K the unknot,*

iii) $\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}$.

□

11.2 Remarks $\nabla_K(z)$ is an 'honest' polynomial in one variable z—no negative powers. ∇_K can be computed by repeated use of the skein relation iii) as in §7.

11.3 Proposition *For a link K*

i) $\nabla_K(z) = \nabla_{rK}(z)$,

ii) $\nabla_K(z) = \nabla_{mK}(-z)$,

iii) *$\nabla_K(z)$ is a polynomial in z^2 if K is a knot.*

Proof i) The relation is unchanged by reversing orientations:

$$\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0}$$

becomes

$$\nabla_{rL_+} - \nabla_{rL_-} = z\nabla_{rL_0}$$

ii) Consider

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z) \qquad 1$$

and

$$\nabla_{mL_-}(z) - \nabla_{mL_+}(z) = z\nabla_{mL_0}(z) \qquad 2$$

Replace z by $-z$ in 2 and get:

$$\nabla_{mL_+}(-z) - \nabla_{mL_-}(-z) = z\nabla_{mL_0}(-z) \qquad 3$$

Since $\nabla_{m(unknot)}(-z) = 1 = \nabla_{unknot}(z)$ using 1 to calculate $\nabla_K(z)$ gives the same result as using 3 to calculate $\nabla_{mK}(-z)$.
iii) unproved.

□

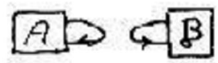
Write $\nabla_K = a_0 + a_1 z + a_2 z^2 + \cdots$

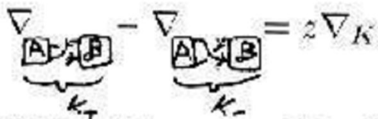
11.4 Proposition

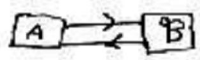
i) $\nabla_K = 0$ *if the link* K *is splittable,*

ii) For a knot K, $a_0 = 1$,

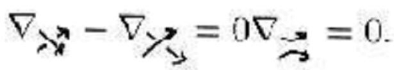
iii) For a 2-component link K, $a_1 = lk(K)$.

Proof i) Choose a diagram for K like this: 
Then



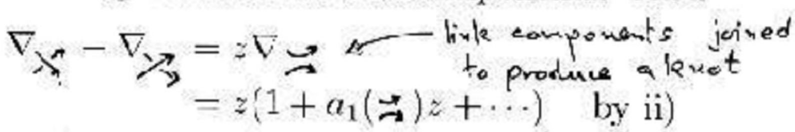
But K_+ and K_- are each equal to 

ii) Put $z = 0$ to get a_0—the skein relation becomes



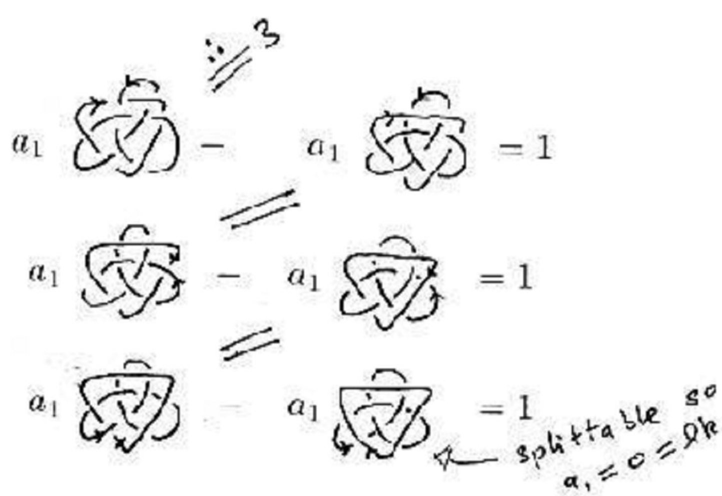
So crossing changes do not affect a_0 and any knot diagram can be changed to a diagram for the unknot by making crossing changes.

iii) Consider a crossing between the two components. Then



So $a_1(L_+) - a_1(L_-) = 1$ (coeff. of z) but $lk(L_+) - lk(L_-) = (x+\frac{1}{2}) - (x-\frac{1}{2}) = 1$. The result follows by induction on the number of crossing changes to get from a link to a splittable link. □

11.5 Example



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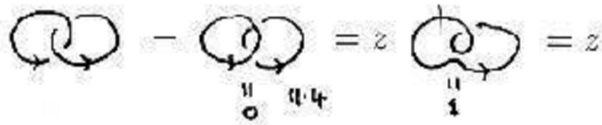
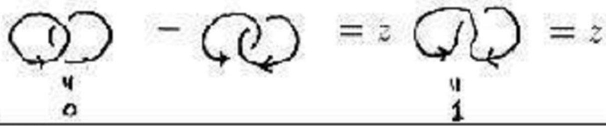
11.6 Examples (omit the symbol '∇')

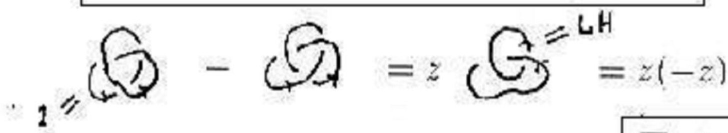
Hopf links





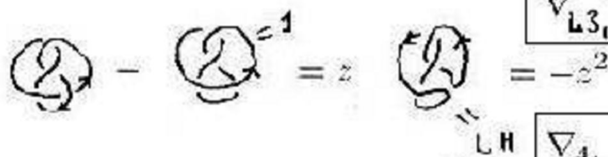
$\nabla_{RightHopf} = z, \qquad \nabla_{LeftHopf} = -z$

$L3_1$



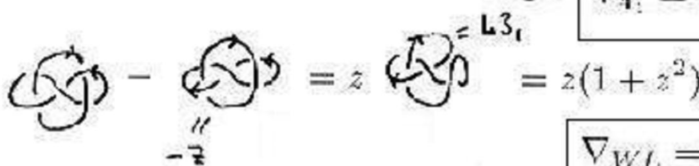
$\nabla_{L3_1} = 1 + z^2$

4_1



$\nabla_{4_1} = 1 - z^2$

Whitehead link



$\nabla_{WL} = z^3$

Recall that the Alexander polynomial was only defined up to multiples of multiplication by $\pm t^{\pm n}$.

11.7 Theorem *For a link K*

$$\nabla_K(x - x^{-1}) = \Delta_K(x^2) \text{ up to } \pm x^{\pm n}$$

□

11.8 Corollary $\det(K) = |\nabla_K(2i)|$.

Proof Put $x = i$ and recall $\det(K) = |\Delta_K(-1)|$. □

11.9 Example For the Whitehead link we have

$$\begin{aligned}\nabla_L = z^3 = (x - x^{-1})^3 &= x^3 - 3x + 3x^{-1} - x^{-3}\\ &= x^4 - 3x^2 + 3 - x^{-2} \text{ up to } \pm x^{\pm n}\\ &= t^2 - 3t + 3 - t^{-1} \text{ if } t = x^2\end{aligned}$$

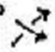
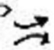
So $\Delta_L \doteq t^2 - 3t + 3 - t^{-1}$.

The HOMFLY (LYMPHTOFU) polynomial

This is a 2-variable generalisation $P(\alpha, z)$ of both Conway and Jones polynomials:

i) K isotopic to L impplies $P_K = P_L$,

ii) $P_K = 1$ if K is the unknot,

iii) αP  $- \alpha^{-1} P$  $= z P$ 

Get Jones by $\alpha = t^{-1}$ and $z = t^{\frac{1}{2}} - t^{\frac{-1}{2}}$, and get Conway by $\alpha = 1$.

11.10 Remark Even the HOMFLY cannot distinguish the Conway knot from the Kinoshita-Terasaka knot and the Conway/Alexander polynomial has value 1 on these knots. See 7.15.

Here is a calculation of the Conway polynomial of the Kinoshita - Teresaka knot.

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