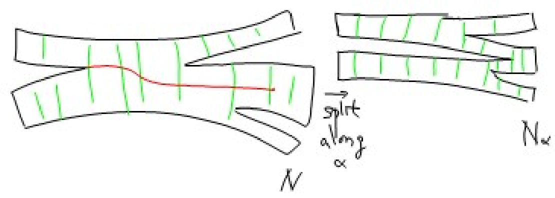
Lecture 4: (I) Splitting tracks τ, $N = N(\tau)$

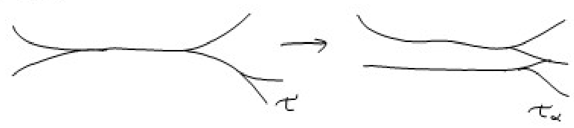
Suppose $\overset{\text{arc}}{\alpha} \subset N$ is transverse to the ties, one end of $\alpha \subset \{\text{cusps}\}$, then call α a splitting arc.



Note that $N(\tau)/_{\text{ties}} \cong \tau$

Def: let $\tau_\alpha := N_\alpha/_{\text{ties}}$.

Example

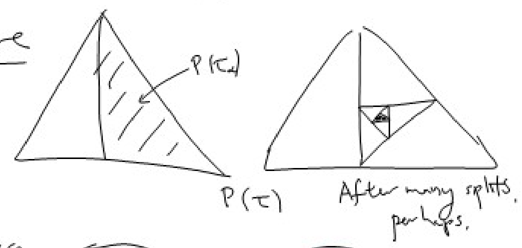


Notice: If $\overset{\text{curve}}{\beta} < \tau_\alpha$ then $\beta < \tau$

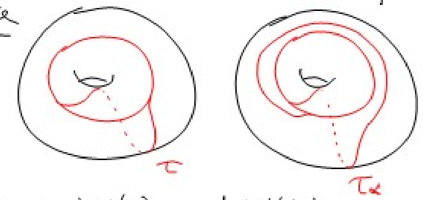
[Since τ_α folds to τ anything τ_α carries so does τ.]

We also have $W(\tau_\alpha) \hookrightarrow W(\tau)$ and
$P(\tau_\alpha) \hookrightarrow P(\tau)$

Picture



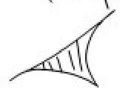
Exercise



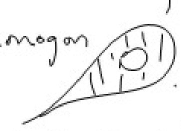
Compute $W(\tau)$ and $W(\tau_\alpha)$.
Check $W(\tau_\alpha) \hookrightarrow W(\tau)$ in advertised fashion.

(II) Basic Observation (Masur Minsky I).

Def: Say $\tau \subset S$ is maximal if every component of $S \setminus N$ is either a triangle



or a once holed monogon

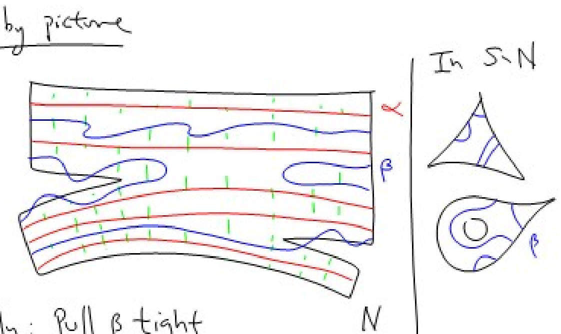


BO: If $\alpha, \beta \in \mathcal{C}(S)$, $d_S(\alpha,\beta) = 1$, τ maximal
$w_\alpha \in \text{interior}(P(\tau))$ [Equiv $w_\alpha(b) > 0$ $\forall b \in B(\tau)$]

Then: $\beta < \tau$.

[Exercise: See Penner Harer]

Pf by picture



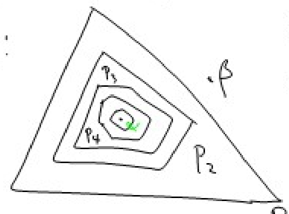
Morally: Pull β tight.

Exercise: Details. //

<u>Corollary of BO</u>: Suppose $\{\tau_i\}_{i=1}^{n}$ is a sequence of max'l tracks s.t. $P_i = P(\tau_i) \subseteq \text{interior}(P_{i-1})$. Suppose further that $\beta \notin P_1$ and $\alpha \in \text{int}(P_n)$.

<u>Then</u> $d_S(\alpha,\beta) \geq n$.

<u>Pf</u>:



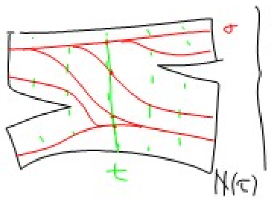
So, If α' is disjoint from α <u>then</u> $\alpha' \in P_n$ <u>so</u> $\alpha' \in \text{int}(P_{n-1})$.
By induction $d_S(\beta,\alpha') \geq n-1$

<u>Thus</u>: $d_S(\alpha,\beta) \geq n$. //

(III) <u>Criterion for nesting</u>.

<u>Def</u>: write $\sigma < \tau$ [both tracks] if $\sigma \subseteq N(\tau)$ and transverse to the ties.

<u>Picture</u>,



So $W(\sigma) \xrightarrow{\rho} W(\tau)$
[ρ injective requires neg index, no smooth ∂'s]

<u>Def</u>: Suppose $w \in W(\sigma)$, $b \in B(\tau)$ then

$$\rho(w)(b) = \sum_{\substack{b' \in B(\sigma) \\ b' \cap t \text{ nonempty},\ t \text{ some fixed tie in } R_b}} w(b')$$

<u>Observe</u> $\rho(w)(b)$ is independent of the choice of t [b/c of the switch conditions.]

<u>Def</u>: If P, Q are polyhedra,

$P \subset Q$ <u>nested</u>

$P \subset \text{int}(Q)$ <u>strictly nested</u>

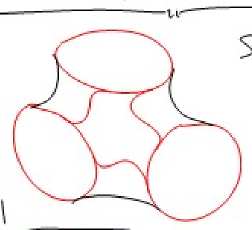
A branch b of τ is

<u>large</u> <u>mixed</u> <u>small</u>.

<u>Def</u>: A track τ is "nice" if..... blah, blah, blah... see PH. [birecurrent]

<u>Penner-Harer</u>: If τ is nice then the face lattice of $W(\tau)$ is isomorphic to the lattice of nice subtracks $\sigma \subseteq \tau$

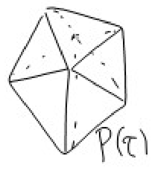
<u>Example</u>:



Standard track in $S_{0,3}$

We may glue these.

Deleting a small branch gives a <u>face</u> of $P(\tau)$.



Criterion: Suppose $\sigma < \tau$ [nice tracks]

and every small branch of σ travels along

every small branch of τ. Then

$$P(\sigma) \subseteq \operatorname{int}(P(\tau)).$$

IV Dimension:

Check: The standard track τ in S_g has

$$|B(\tau)| = 18(g-1)$$

$$|\mathcal{S}(\tau)| = 12(g-1).$$

Exercise: $\dim(W(\tau)) = 6g-6 = 2\xi(S)$

Exercise: Do all of this for $S_{g,n}$

That is the switch conditions are independent.

Hard exercise: $\dim(W(\tau_\alpha)) < \dim(W(\tau))$,

if $\alpha < \tau$ is a carried arc.