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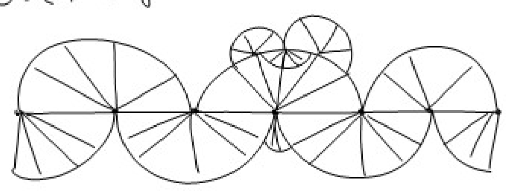
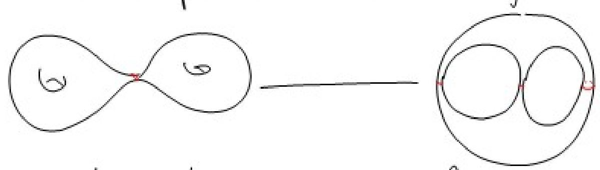
Hierarchies and the curve complex

Outline of course



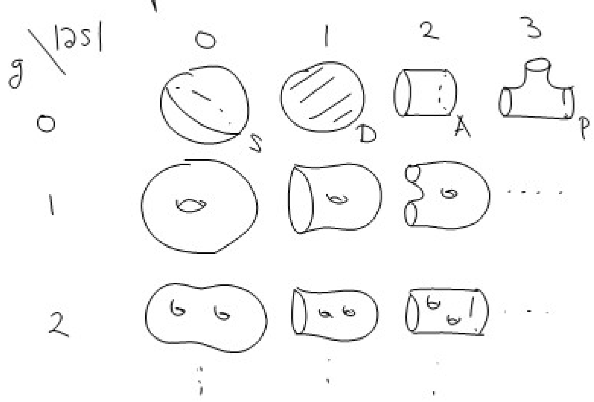
Source paper is Masur Minsky II [2000]

Very roughly: A hierarchy records how the "shape" of a surface changes



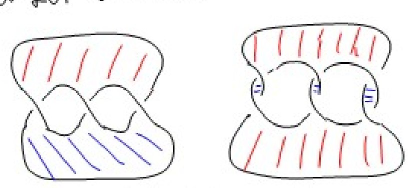
The hierarchy is a system of tight geodesics in the complex of curves, controlled by subsurface projections. Applications include a distance estimate for the mapping class group and etc [ELC]

(1a) Surfaces. Classification of compact orientable, connected surfaces



Very powerful theorem.

Exercise:



Identify surfaces.

Exercise: Redo classification including non-orient.

Isotopies: $\mathrm{Homeo}(S) = \{ f : S \to S \mid \text{homeo} \}$

Write $f \sim g$ (f isotopic to g) if

$\exists$ map $F : S \times I \to S$ s.t.

1) $F_0 = f$, $F_1 = g$
2) $F_t \in \mathrm{Homeo}(S)$, $\forall t \in I$.

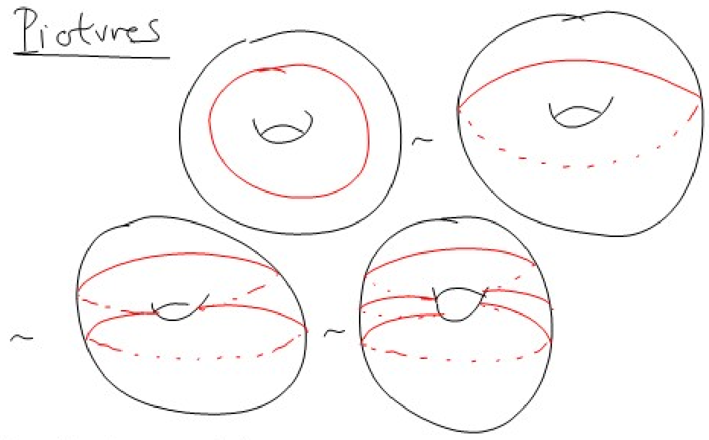
Here $I = [0,1]$, $F_t(x) = F(x,t)$

Define $\mathrm{Homeo}_0(S) \equiv \left\{ f \in \mathrm{Homeo}(S) \mid f \sim \mathrm{Id}_S \right\}$

Exercise: $\mathrm{Homeo}_0(S) \lhd \mathrm{Homeo}(S)$.

If $X, Y \subseteq S$ write $X \sim Y$ if $\exists f$ in $\mathrm{Homeo}_0(S)$ s.t. $Y = f(X)$.

Pictures



Def. $MCG(S)$ = mapping class group
$$= \frac{\mathrm{Homeo}(S)}{\mathrm{Homeo}_0(S)} = \frac{\mathrm{Homeo}(S)}{\sim}$$

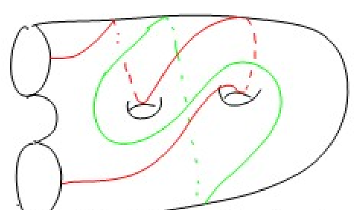
Exercise: $MCG(S) \neq \{1\}$.

[$\mathrm{Homeo}^+(S) = \{\text{orient. preserving } f \in \mathrm{Homeo}(S)\}$
a) Show $\mathrm{Homeo}_0(S) \triangleleft \mathrm{Homeo}^+(S)$
b) Show $MCG^+(S) \neq \{1\}$ if $S \neq S^2, D^2$]

[Further definitions $MCG(S, \partial S)$, $MCG^{\partial}(S)$ --]

(III) Curves and Arcs

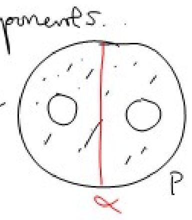
A curve $\alpha \subset S$ is (the image of) an embedding $S^1 \hookrightarrow S$. An arc $\alpha \subset S$ is (—) a proper embedding $I \hookrightarrow S$.



[Question: Do we care about parametrization?
Ans: Mostly no; sometimes think about oriented $\alpha \subset S$

Def. Say $\alpha \subset S$ is separating if $S \setminus \alpha$ has two components.

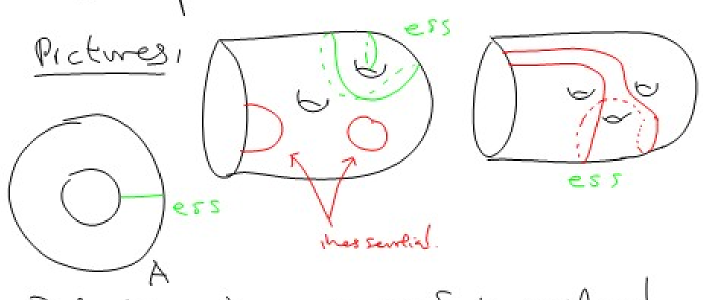
[NB Arcs can separate



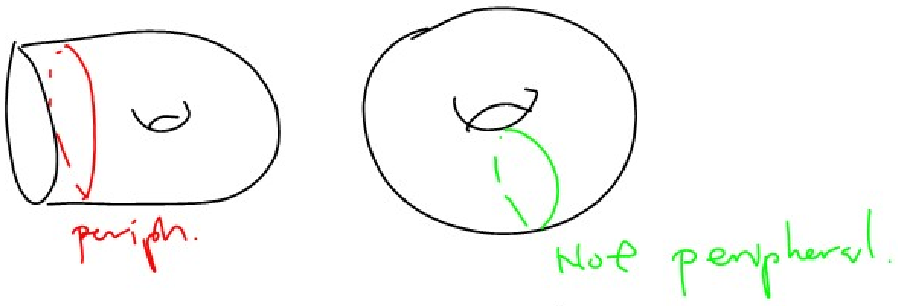
]

Def. Say $\alpha \subset S$ is inessential if α separates and $S \setminus \alpha$ has a component with closure a disk.

Pictures:



Definition: A curve $\alpha \subset S$ is peripheral (boundary parallel) if α separates and $S \setminus \alpha$ has a component with closure an annulus.



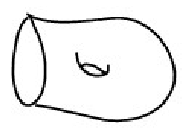
<u>Define</u>: $\mathcal{C}^0(S) = \{\text{essential, nonperip curves}\}/\sim$

$\mathcal{A}^0(S) = \{\text{ess. arcs}\}/\sim$

$\mathcal{AC}^0(S) = \mathcal{A}(S) \cup \mathcal{C}(S)$.

<u>Exercise</u>: Compute $\mathcal{AC}(S)$ for $S = S^2, D, A, P$
$= \mathbb{T}^2, S_{1,1}$

<u>Exercise</u>: Find natural maps



$\mathcal{A}(S_{1,1}) \longrightarrow \mathcal{C}(S_{1,1})$ 1 to 1

$\mathcal{A}(S_{0,4}) \longrightarrow \mathcal{C}(S_{0,4})$ 6 to 1.

<u>Exercise</u>: Show $\mathcal{AC}(S)$ is countable.
Show $\mathcal{AC}(S)$ infinite if $S \neq S^2, D^2, A, P$.

<u>Exercise</u>: Compute $\mathcal{AC}(S)/MCG(S)$.