# Lecture III.

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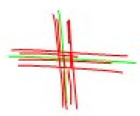
with date, name, statement of problem.

② "Rotate by 90° in def of transversality



} $N(\alpha), N(\beta)$ have product structures $\alpha \times I$, $\beta \times I$: 1st and 2nd coords swap places in the square.

③ Quantifiers simplify to $\exists N(\alpha), N(\alpha), N(\beta)$."

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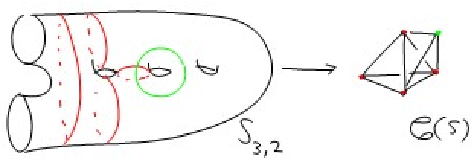
## (I) Arc and curve complexes [Harvey]

Def: $\mathcal{AC}(S)$ is the simplical complex with vertex set $\mathcal{AC}^0(S)$ and multicurves give simplices. $\mathcal{A}(S)$, $\mathcal{C}(S)$ are the subcomplexes spanned by $\mathcal{A}^0(S)$ and $\mathcal{C}^0(S)$ respectively.

Picture




Def: $\xi(S) = 3g - 3 + n$

$= \text{complexity of } S (\cong S_{g,n})$

Exercise: $\xi(S) - 1 = \text{dimension of } \mathcal{C}(S)$.

Def: Say $X \subseteq S$ is a cleanly embedded subsurface if every component of $\partial X \setminus \partial S$ is essential, non peripheral. If $X \neq S$ call $X$ strict. Note [Prove] if $X \subseteq S$ is strict then $\xi(X) < \xi(S)$. [Thus may induct on $\xi$].

Exercise: $\mathcal{A}(S)$ is connected.
$\mathcal{C}(S)$ is connected, $\mathcal{AC}$ as well.
[Supposing that $\xi(S) \geq 2$.]

Exercise: If $\alpha \in \mathcal{C}^0(S)$ then

$\mathrm{lk}_e(\alpha) \cong \mathcal{C}(S \setminus \alpha)$.

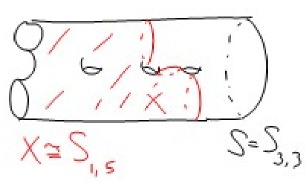
[$\mathcal{C}(X) * \mathcal{C}(Y)$ if $X \sqcup Y = S \setminus \alpha$
i.e. if $\alpha$ separates.]

Prop: $\mathcal{C}(S)$ is not locally finite.

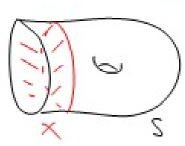
Pf: Exercises above. //

Picture

$X \subseteq S$ cleanly emb.




Not cleanly emb.



II **Distance**: Suppose $K$ is a simp. comp.

Define $d_K : K^\circ \times K^\circ \to \mathbb{N} \cup \{\infty\}$

$$d_K(\alpha,\beta) = \min \left\{ \underbrace{|P|}_{\#\text{ of edges.}} : P \text{ edge path connecting } \alpha \text{ to } \beta \right\}$$

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Use $d_{\mathcal{A}(S)}$, $d_{\mathcal{AC}(S)}$ as above but abbrev.

$d_{\mathcal{C}(S)} = d_S$.

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Exercise: If $\alpha, \beta \in \mathcal{C}^\circ(S)$ then

$$d_S(\alpha,\beta) = d_{\mathcal{AC}}(\alpha,\beta) \quad \text{i.e.}$$

The inclusion $\mathcal{C}(S) \hookrightarrow \mathcal{AC}(S)$ is an isometric embedding. $[\xi(S) \geq 2]$.

[Rmk: $d_S(\alpha,\beta) < \infty$ by exercise $(\xi(S) \geq 2)$]

Challange: The inclusion $\mathcal{A}(S) \hookrightarrow \mathcal{AC}(S)$ is not even a quasi-isometric embedding.

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Theorem [Kobayashi] $\operatorname{diam}_{\mathcal{C}}(\mathcal{C}(S)) = \infty$.

Theorem [MM I] $\mathcal{C}(S)$ is Gromov hyperbolic.

Exercise: Look-up def of hyperbolicity.

Exercise: The inclusion $\mathcal{C}(S) \hookrightarrow \mathcal{AC}(S)$ is a quasi-isometry.

Definition: A relation $f : (X, d_X) \to (Y, d_Y)$ is a $K$ quasi isometric embedding if

$\forall x, y \in X$, $\bar{x}, \bar{y} \in f(x), f(y)$

$$\frac{1}{K}\left(d_Y(\bar{x},\bar{y}) - K\right) \leq d_X(x,y) \leq K\, d_Y(\bar{x},\bar{y}) + K$$

if additionally $Y = N_K(f(X))$ then call $f$ a quasi-isometry.

Exercise: Show: $f$ is quasi-isometry iff $\exists g : Y \to X$ a quasi-isom. inverse

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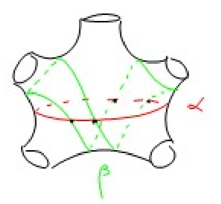
Def: $\alpha, \beta \subseteq S$ fill iff $d_S(\alpha,\beta) \geq 3$.




$d_S(\alpha,\beta) = 3$ in this case.

Exercise: Find $\alpha, \beta \subset S_{0,5}$ s.t. $d_S(\alpha,\beta) = 4$.

Exercise: find dist 3 curves in $S = S_2$.

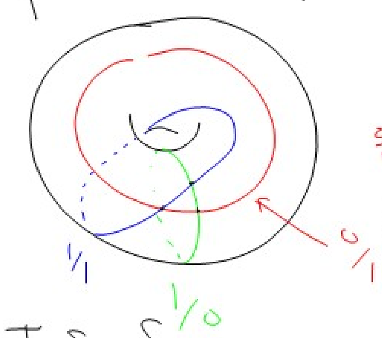
(III) Low complexity:

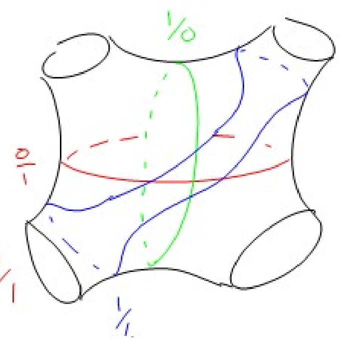
for $S = T, S_{1,1}, S_{0,4}$, $\mathcal{C}(S)$ disconnected.

So redefine $\mathcal{C}(S)$ as follows.

- $S = \mathbb{T}^2$ or $S_{1,1}$ simplices $\Delta \subset \mathcal{C}^\circ(S)$ if $\forall \alpha, \beta \in \Delta$ $i(\alpha,\beta) = 1$.
- $S = S_{0,4}$: $\Delta \subseteq \mathcal{C}^\circ(S)$ is a simplex if $\forall \alpha, \beta \in \Delta$ $i(\alpha,\beta) = 2$.

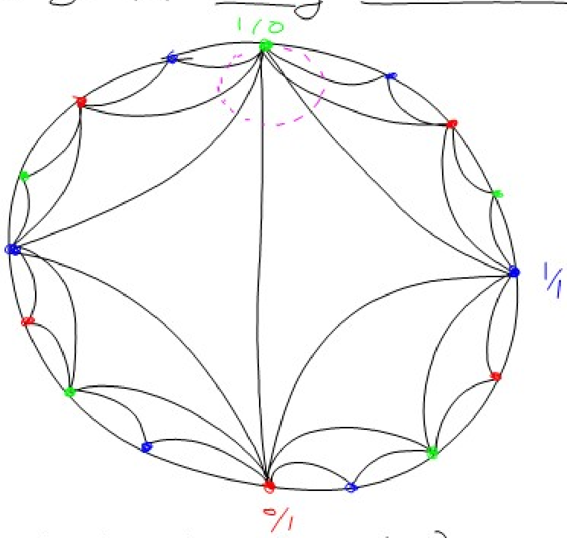
Pictures:



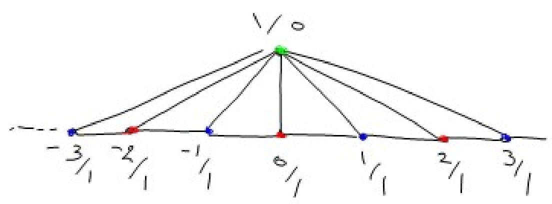




So for $S = T, S_{1,1}, S_{0,4}$

$\mathcal{C}(S) \cong \mathcal{F}$ the Farey tessellation.




Notice that the link $\mathrm{lk}(1/0) \cong \mathbb{Z}$




} curves of integer slope in $\mathbb{T}^2$.