Lecture V:

$$\begin{array}{ccc} \mathcal{AC}(S) & \xrightarrow{\pi_X} & \mathcal{C}(X) \\ & {\scriptstyle \kappa_X}\searrow \quad \nearrow{\scriptstyle \sigma_X} & \\ & \mathcal{AC}(X) & \end{array}$$

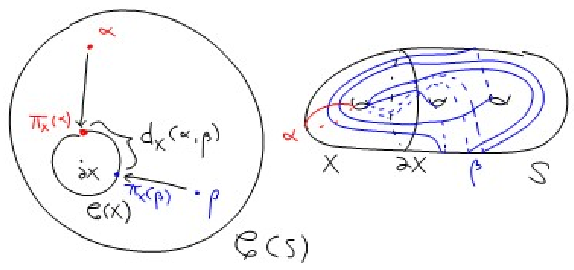
This defines the subsurface projection relation.

[first considered by N. Ivanov. We'll follow Masur-Minsky II].

Def: If $\alpha, \beta \in \mathcal{AC}(S)$ and both cut X

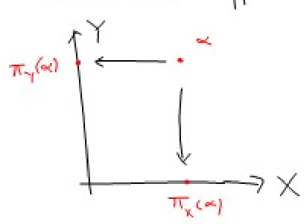
$$d_X(\alpha, \beta) := \operatorname{diam}_{\mathcal{C}(X)} \left(\pi_X(\alpha) \cup \pi_X(\beta) \right)$$

is the sub. projection distance.



Very simple motivating examples of projections.

"Disjoint": Suppose X, Y are the x, y axes in $\mathbb{R}^2$



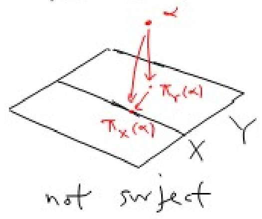
Suppose π_X, π_Y are closest point projections to X, Y

Note that

$$\mathbb{R}^2 \xrightarrow[\pi_X \times \pi_Y]{} X \times Y \text{ surjects}$$

Eg cannot recover $\pi_X(\alpha)$ from $\pi_Y(\alpha)$.

"Nested" Suppose $X^1 \subseteq Y^2 \subseteq \mathbb{R}^3$ are the x axis and the xy plane



Thus:

$$\pi_X(\alpha) = \pi_X \pi_Y(\alpha)$$

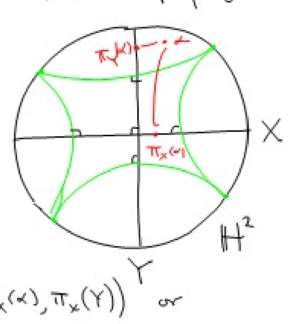
So $\mathbb{R}^3 \xrightarrow[\pi_X \times \pi_Y]{} X \times Y$ does not surject

$$(x, y, z) \longmapsto (x, x, y)$$

instead we hit the graph of $\pi_X : Y \to X$.

"Overlap" Suppose X, Y are perp. geodesics in $\mathbb{H}^2$

Let d_X, d_Y be distance in X, Y



$\forall \alpha \in \mathbb{H}^2$

Exercise: either

$$d_X(\alpha, Y) = d_X(\pi_X(\alpha), \pi_X(Y)) \text{ or}$$

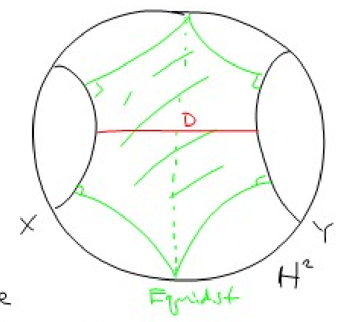
$$d_Y(X, \alpha) \leq \ln(1+\sqrt{2})$$ [or both if α is inside the green rectangle.]

So: The map $\mathbb{H}^2 \xrightarrow[\pi_X \times \pi_Y]{} X \times Y$ has image landing in a $\ln(1+\sqrt{2})$-neighborhood of $X \times \{X \cap Y\} \cup \{X \cap Y\} \times Y \subseteq X \times Y$.

[Sup metric.]

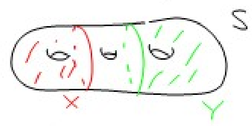
One more picture

π_X, π_Y behave as in the overlapping case. Lower bounds on D give upper bound on "lack of motion".



Back to surfaces: Suppose $X, Y \subsetneq S$ are cleanly embedded. $X \cap Y$ is minimal.

Disjoint:



Exercise: Show that $\pi_X \times \pi_Y : \mathcal{AC}(S) \to \mathcal{C}(X) \times \mathcal{C}(Y)$ coarsely surjects.

Perhaps useful to show that.
$\sigma_X | \mathcal{A}(X) : \mathcal{A}(X) \longrightarrow \mathcal{C}(X)$ is coarsely surjective.

Nested: Suppose $X \subset Y$ is cleanly emb, [So X not perip. annulus] and $X \neq S_{0,3}$

Lemma: $\exists$ constant M_1 (uniform) so that:
$\forall \alpha \in \mathcal{AC}(S)$ if α cuts X [$\pi_X(\alpha) \neq \emptyset$]
Then $d_X(\alpha, \pi_Y(\alpha)) \leq M_1$.

[ie $\operatorname{diam}_X(\pi_X(\alpha), \pi_X\pi_Y(\alpha)) \leq M_1$ ie coarsely $\pi_X(\alpha) = \pi_X \pi_Y(\alpha)$]

[**Exercise**: Check $\forall \alpha \in \mathcal{AC}(S)$ if $\pi_X(\alpha) \neq \emptyset$ then $\operatorname{diam}_X(\pi_X(\alpha))$ unif bounded]

Pf of Lemma as series of exercises.

Exercise: $\sigma_X : \mathcal{AC}(X) \longrightarrow \mathcal{C}(X)$ is a quasi-isometry

[in fact it is a quasi inverse for the inclusion $\mathcal{C}(X) \hookrightarrow \mathcal{AC}(X)$]

[**NB**: This is *false* for $\sigma_X | \mathcal{A}(X) : \mathcal{A}(X) \to \mathcal{C}(X)$]

Since $\pi_X = \sigma_X \circ \kappa_X$ and $\pi_X \circ \pi_Y = \sigma_X \circ \kappa_X \circ \sigma_Y \circ \kappa_Y$

and σ_X has an quasi inverse; It suffices to bound $\operatorname{diam}_{\mathcal{AC}(X)}(\kappa_X(\alpha) \cup \kappa_X \sigma_Y \kappa_Y(\alpha))$.

Next: **Exercise**: $\kappa_X(\alpha) = \kappa_X \kappa_Y(\alpha)$
[**Hint**: $S^X = (S^Y)^X$]

Define $A = \kappa_Y(\alpha)$. So suffices to bound $\operatorname{diam}_{\mathcal{AC}(X)}(\kappa_X(A) \cup \kappa_X \sigma_Y(A))$

If $A = \{\alpha\}$ is a curve we are done
So pick $\beta \in A \subseteq \mathcal{A}(Y)$. Let δ = compt's of ∂Y meeting β. $N := N(\beta \cup \delta)$ since 1) X cuts N 2) $X \not\subset N$ we are done. // details.