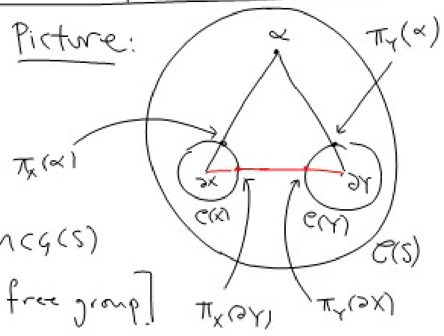
Lecture VI :

(I) **Behrstock's Lemma**: Suppose $X, Y \subseteq S$ overlap: ∂X cuts Y and ∂Y cuts X. $\exists$ constant $M = M(S)$ s.t. $\forall \alpha \in \mathcal{AC}(S)$ if α cuts X, Y then one of

$d_X(\alpha, \partial Y)$ or $d_Y(\partial X, \alpha)$ is at most $M(S)$.

Picture:



Exercise

(1) Use this to embed $\mathbb{F}_2 \to MCG(S)$

[$\mathbb{F}_2 = \langle a, b \rangle$ free group]

(2) Now embed $\mathbb{Z}^2 * \mathbb{Z}$, $\mathbb{F}_3$ (easy!). $\mathbb{F}_k$, $\mathbb{F}_2 \times \mathbb{F}_2$, $\Gamma(\pentagon) =$ right angled Artin group... etc..

[Hint: Available on request.]

We will use:

Hempel's Lemma: Suppose $\alpha, \beta \in \mathcal{C}(S)$, $i(\alpha, \beta) \neq 0$: Then

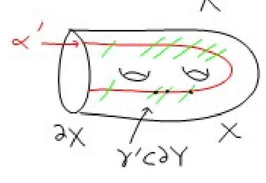
$$d_S(\alpha, \beta) \leq 2 \log_2 i(\alpha, \beta) + 2$$

Exercise: Find a linear bound, or any bound at all, really. [Question: Special cases are $S_{0,4}$ and $S_{1,1}$. Non orient. surfaces?]

Exercise: The inequality cannot be reversed

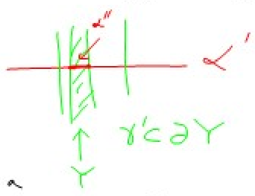
Pf of Behrstock: Suppose $d_X(\alpha, \partial Y) > M$

Thus $i_X(\alpha, \partial Y) \equiv i(\kappa_X(\alpha), \kappa_X(\partial Y)) \geq 3$.



Better $\exists$ components $\alpha' \in \kappa_X(\alpha)$, $\gamma' \in \kappa_X(\partial Y)$ s.t. $i_X(\alpha', \gamma') \geq 3$.

So we see



So $\kappa_Y(\alpha')$ has a component $\alpha'' \subset \alpha'$. So: $\alpha'' \cap \partial X = \emptyset$.

Thus: $d_{\mathcal{AC}(Y)}(\kappa_Y(\alpha), \kappa_Y(\partial X)) =$

$\text{diam}_{\mathcal{AC}(Y)}(\kappa_Y(\alpha) \cup \kappa_Y(\partial X))$

but $\text{diam}_{\mathcal{AC}}(\{\alpha''\} \cup \kappa_Y(\partial X)) \leq 2$

Since $\text{diam}_{\mathcal{AC}(Y)}(\kappa_Y(\alpha)) \leq 1$, we may surger into $\mathcal{C}(Y)$ and find

$d_Y(\alpha, \partial X) \leq M$. //

Exercise: X or Y $= \mathbb{A}^2$?

II) Markings:

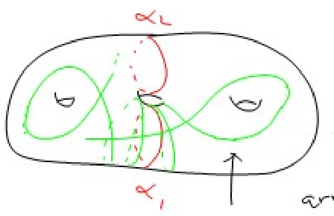
A <u>marking</u> $\mu = \left(\{\alpha_i\}_{i\in I}, \{\tau_i\}_{i\in J\subseteq I}\right)$

has <u>base curves</u> $\text{base}(\mu) = \{\alpha_i\}$

a multicurve in $\mathcal{C}(S)$, and <u>transversals</u>

$\text{trans}(\mu) = \{\tau_i\}$ s.t. $\tau_i \in \mathcal{C}(\alpha_i)$
$= \mathcal{C}(N(\alpha_i))$

<u>Picture</u>



i.e. τ_i is an arc in S^{α_i}

$\rho_{\alpha_i}(\tau_i)$

[$\rho_{\alpha_i}: S^{\alpha_i} \to S$ is the covering map.

<u>Def</u>: μ is <u>complete</u> if $\text{base}(\mu)$ is a <u>pants decomposition</u> $\left[\,|\text{base}(\mu)| = \xi(S)\,\right]$ and each α_i has a transversal τ_i.

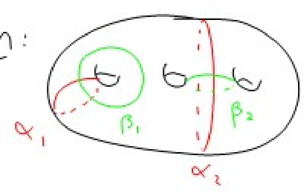
<u>Ex</u>:

<u>Def</u>: A marking is <u>clean</u> if $\forall i$ $\beta_i = \rho_{\alpha_i}(\tau_i)$ is a simple closed curve and

$$(*)\quad i(\beta_i, \alpha_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \quad \alpha_i \text{ nonsep} \\ 2 & i = j \quad \alpha_i \text{ sep} \end{cases} \text{ in } X_i$$

<u>Def</u>: $X_i \subseteq S \setminus (\text{base}(\mu) \setminus \{\alpha_i\})$ is the component containing α_i

<u>Clean</u>:



<u>Observe</u>: If μ is complete then $\forall i$ $X_i \cong S_{0,4}$ or $S_{1,1}$.

<u>Def</u>: $\mathcal{M}^\circ(S) = \{\text{complete, clean markings}\}$ (up to isotopy)

<u>Note</u>: If $S = S_{0,4}$ or $S_{1,1}$ then $\mathcal{M}^\circ(S) \cong$ oriented edges in $\mathcal{F}$.

<u>Finitness Lemma</u>: $\mathcal{M}^\circ(S)/_{MCG(S)}$ is finite. <u>Pf</u>: Exercise.

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