Lecture VIII $\mathcal{MCG}(S)$ and $\mathcal{M}(S)$.

$(X, d_X) \xrightarrow{f} (Y, d_Y)$ is an L-quasi-isometry

1) $f(X)$ is L-dense in Y

   $[\forall y \in Y\ \exists x \in X\ \ d_Y(f(x), y) \leq L]$

2) $\forall x, y \in X\ \ d_X(x,y) =_L d_Y(f(x), f(y))$.

   [i.e. f is an L-quasi-isom. emb.]

Recall $\forall L, a, b \in \mathbb{R}_{\geq 0}$ $L \geq 1$

- $a =_L b$ if $a \leq_L b$ and $b \leq_L a$
- $a \leq_L b$ if $a \leq L \cdot b + L$.

Exercise: $\mathbb{Z} \underset{q.i.}{\cong} \mathbb{R}$, $\mathbb{Z}^2 \cong \mathbb{R}^2$.

*) $SL(2, \mathbb{Z}) \cong F_2 \cong T_3$. 3-valent tree.

*) $\mathcal{F} \cong T_\infty$ ($\mathbb{Z}$-valent tree).

*) $T_3 \not\cong \mathcal{F}$ the Farey graph.

Fix any $\mathbb{X} \subseteq \mathcal{MCG}(S)$ finite generating set.

Let $d_{\mathbb{X}}(g,h) = |g^{-1} \cdot h|_{\mathbb{X}}$ be the word metric.

To define this, let $\Gamma(\mathbb{X})$ be the graph with vertex set $\mathcal{MCG}(S)$ and edges $\{(g,h) \mid \exists x \in \mathbb{X}, h = gx\}$. [Right Cayley Graph]

Rmk: $\mathcal{MCG}(S)$ act on $\Gamma_{\mathbb{X}}$, via isometries, on the left because: if $g, h, f \in \mathcal{MCG}(S)$ then $h = gx \iff fh = fgx$.

Exercise: Compute the Cayley graph of Sym_3 (Sym_4?) as generated by $(12), (23)$.

Def: $|g|_{\mathbb{X}} = d_\Gamma(Id, g)$. Thus

$$d_\Gamma(g,h) = d_\Gamma(Id, g^{-1} \cdot h) = |g^{-1} \cdot h|_{\mathbb{X}}.$$

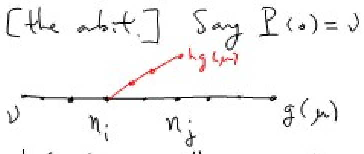
Thm: Fix $\mu \in \mathcal{M}^\circ(S)$. The orbit map

$$\begin{array}{ccc} \mathcal{MCG}(S) & \longrightarrow & \mathcal{M}(S) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \longmapsto & g(\mu) \end{array}$$

is a quasi-isometry between the word metric $d_{\mathbb{X}}$ and the "elementary move" metric.

Rmk: The constants are sensitive to the choice of $S = S_{g,n}$ and of $\mathbb{X} \subseteq \mathcal{MCG}(S)$ [and to how carefully we clean after flipping] and insensitive to the choice of μ.

Pf: Density: Fix $\nu \in \mathcal{M}(S)$ [want $g \in MCG$ s.t. $d_{\mathcal{M}}(\nu, g(\mu))$ unif. bounded.] By connectedness choose a shortest path $\mathcal{P}$ from ν to $MCG(S) \cdot \mu$ [the orbit.] Say $\mathcal{P}(0) = \nu$, $\mathcal{P}(n) = g(\mu)$.



If $\exists\, i < j$ s.t. $h(\eta_j) = \eta_i$ then consider

$Q = \mathcal{P}[0, i] \cup h(\mathcal{P}[j, n])$ $[h \in MCG]$

and $|Q| < |\mathcal{P}|$ a contradiction.

Thus $|\mathcal{P}| \leq \left| \mathcal{M}^0(S) / MCG(S) \right|$ And apply finiteness //

$d_{\mathcal{M}} \leq_L d_{\mathbb{X}}$: Fix $g, h \in MCG(S)$

So $d_{\mathcal{M}}(g(\mu), h(\mu)) = d_{\mathcal{M}}(\mu, g^{-1}h(\mu))$

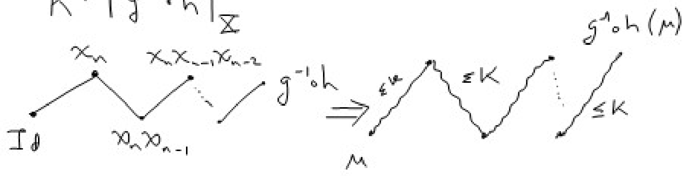
Let $K = \max\{ d_{\mathcal{M}}(\mu, x(\mu)) \mid x \in \mathbb{X} \}$ [Connectedness + finite generation]

Write $g^{-1} \circ h = \prod_{j=1}^{n} x_j$

Thus: $d_{\mathcal{M}}\left(\prod_{j=i+1}^{n} x_j(\mu), \prod_{j=i}^{n} x_j(\mu) \right) \leq K$

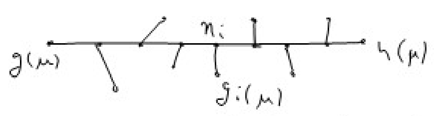
$\left[= d_{\mathcal{M}}(\mu, x_i(\mu)) \right)$

$\Rightarrow$ by triangle inequality $d_{\mathcal{M}}(\mu, g^{-1}h\,\mu) \leq K \cdot |g^{-1} \circ h|_{\mathbb{X}}$



$d_{\mathbb{X}} \leq_L d_{\mathcal{M}}$. Fix $g, h \in MCG$. Pick geodesic $\mathcal{P} \subseteq \mathcal{M}(S)$ from $g(\mu)$ to $h(\mu)$.

By density $\forall\, \eta_i \in \mathcal{P}$ pick $g_i \in MCG(S)$ s.t. $d_{\mathcal{M}}(\eta_i, g_i(\mu)) \leq K = |\mathcal{M}^0 / MCG|$



Claim: $\exists\, L$ s.t. $|g_i^{-1} \circ g_{i+1}|_{\mathbb{X}} \leq L$

[L independent of i or $\mathcal{P}$ or g or h]

Pf: $B_{\mathcal{M}}(\mu, 2K+1)$ = ball of radius $2K+1$ has at most C^{2K+1} elements.

[because only $O(\xi(S))$ elem. moves.]

Also, $\exists$ bound on $|g|_{\mathbb{X}}$ as g varies in $\mathrm{Stab}(\eta)$ and $\eta \in \mathcal{M}^0(S)$.

$\rightarrow \exists$ uniform L s.t. $|g^{-1} \circ h|_{\mathbb{X}} \leq L|\mathcal{P}| + L$. //

[Additive error required as $g(\mu) = h(\mu) \not\Rightarrow g = h$.]

Rmk: $|\mathrm{Stab}(\mu)| \leq 84(g-1)$ $[n = 0]$