Lecture IX:

(I) The distance estimate

If $\mu, \nu \in \mathcal{M}(S)$ define $d_X(\mu,\nu) \equiv \operatorname{diam}_X(\pi_X(\mu) \cup \pi_X(\nu))$.

Also If $x, C \in \mathbb{R}_{\geq 0}$ define

$$[x]_C = \begin{cases} 0, & x < C \\ x, & x \geq C \end{cases}$$

Thm [Masur-Minsky]

$\forall S = S_{g,n}\ \exists C_0 = C_0(S)\ \forall C \geq C_0\ \exists A \geq 1$
$\forall \mu, \nu \in \mathcal{M}(S)$

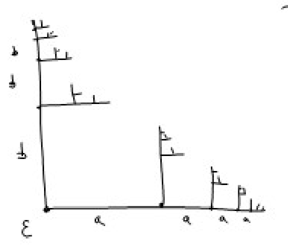
$$d_{\mathcal{M}}(\mu,\nu) =_A \sum_{X \subseteq S} [d_X(\mu,\nu)]_C$$

Rmk. As $MCG(S) \underset{\text{qi}}{\cong} \mathcal{M}(S)$ we have the same result for $MCG(S)$.

Easier examples:

1) Exercise. Show that the L^2 and L^1 metrics on $\mathbb{R}^2$ are bilipschitz.

2) Let $\Sigma = \{a, b\}$. let $\Sigma^* = \{\text{words over } \Sigma\}$



The rooted infinite binary tree.

$|w|$ = length of w = # of letters

$|w|_s$ = # of syllables

$w = aaabbaabaabb$, $|w| = 12$, $|w|_s = 6$

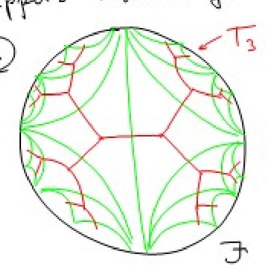
Let $|w|_i$ = length of the i^{th} syllable.

Exercise: Show for all $C \geq C_0 = 0\ \exists A \geq 1$ s.t. $\forall w \in \Sigma^*$,

$$|w| =_A [|w|_s]_C + \sum_{i \in \mathbb{N}} [|w|_i]_C$$

Notice that this example appears inside of

(3)



Here T_3 is the dual tree to $\mathcal{F}$. Exercise

$$GL(2,\mathbb{Z}) \cong MCG(\mathbb{T}^2) \underset{\text{qi}}{\cong} T_3.$$

Here: If $u, v \in T_3$ then

$$d_{T_3}(u,v) =_A \underbrace{[d_{\mathcal{F}}(u,v)]_C}_{S = S_{1,1}} + \underbrace{\sum_{r \in \mathbb{Q} \cup \{\infty\}} [d_r(u,v)]_C}_{\text{annuli}}$$

(II) Projection bounds

Lemma: Suppose $X \subseteq S$ is essential, cleanly embedded. Suppose $\alpha, \beta \in \mathcal{AC}(S)$ cut X, $d_S(\alpha,\beta) = 1$. Then

$$\operatorname{diam}_X(\kappa_X(\alpha) \cup \kappa_X(\beta)) \le 1$$

$\Big[\le 3$ if $X = \mathbb{A}^2$ and $\xi(S) = 1$.

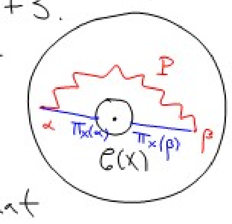
Pf: Exercise.

Corollary: So $d_X(\alpha,\beta) \le 3$. [Lemma 2.2 MM II]

Corollary: If $P \subseteq \mathcal{AC}(S)$ is a path s.t. $\forall \alpha_i \in P$, α_i cuts X then if $\alpha = \alpha_0$, $\beta = \alpha_N$

Then $d_X(\alpha,\beta) \le 3|P| + 3$.

This is called "Lipschitz projection to subsurfaces"



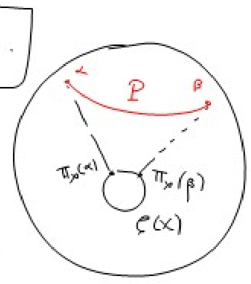
Exercise: The hypothesis that all α_i cut X is crucial.

Exercise: Prove, without laminations etc, $\forall S$ (if $\xi(S) \ge 1$ or $S \cong \mathbb{A}^2$) that $\operatorname{diam}(\mathcal{C}(S)) = \infty$.

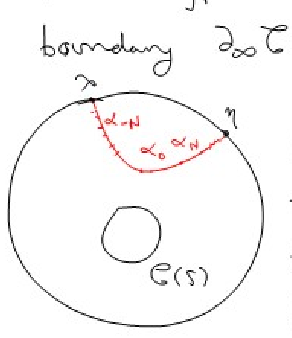
[**Rmk**: We are mostly ignoring $S = S_{0,3}$]

(III) For geodesics $P \subseteq \mathcal{C}(S)$ a stronger result holds

Bounded Geodesic Image Theorem: $\exists M = M(S)$, for any $P \subseteq \mathcal{C}(S)$ geodesic connecting α, β s.t. $\forall \alpha_i \in P$, α_i cuts X we have $\boxed{d_X(\alpha,\beta) \le M.}$



Idea: Following [Klarreich] and [Masur Minsky I] we know that $\mathcal{C}(S)$ is Gromov hyperbolic so has a Gromov boundary $\partial_\infty \mathcal{C}(S) \cong \mathcal{EL}(S)$ [ending laminations]



As $i \to \pm\infty$ we see that $\kappa_X(\alpha_i)$ converges to λ, η in sense of Hausdorff, and they do so in bounded time. //Idea.

Exercise: Find a proof of this following Bowditch

Hint: