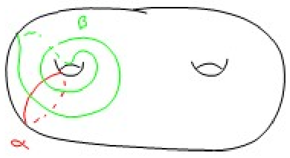
Lecture X.

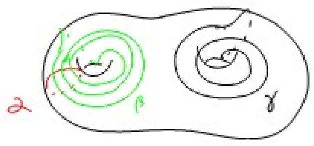
(I) Tight geodesics:



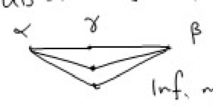
<u>Def</u> $X(\alpha,\beta) =$ the <u>support</u> of α and β: This is the subsurface obtained by

1) Make α, β tight $(|\alpha \cap \beta| = i(\alpha,\beta))$
2) Let $N = N(\alpha \cup \beta)$ (reg neighborhood)
3) Add to N all disks and <u>peripheral</u> annuli of $S \setminus N$, to get X.

So $X(\alpha,\beta)$ is the smallest (up to isotopy) cleanly embedded surface in S containing α and β.



Note that α, β have distance 2 in $\mathcal{C}(S)$

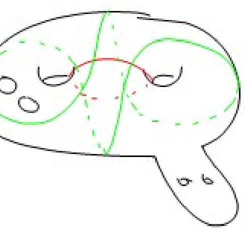


Inf. many geodesics

<u>Notice</u>: If $d_S(\alpha,\beta) = 2$ then

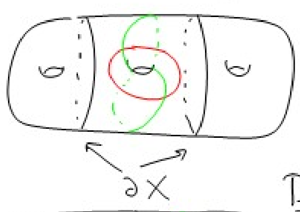
α —— $\partial_S X(\alpha,\beta)$ —— β } is "canonical" choice of geodesic.

<u>Issue</u>: $\partial_S X(\alpha,\beta)$ need not be connected!

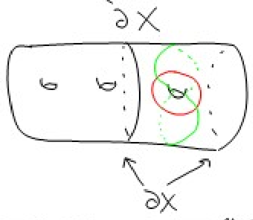


α, β fill the surface S_0 connect sum with stuff to get S.

<u>Better</u>:



<u>Def</u>: $\partial_S X = \partial X - \partial S$.

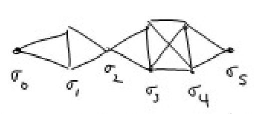


<u>Def</u>: $g = \{\sigma_i\}$ $\left[\begin{array}{l}\text{indexed by } [M,N] \subseteq \mathbb{Z} \\ \text{or by } [M,\infty) \subseteq \mathbb{Z} \\ \text{or } (-\infty, N] \subseteq \mathbb{Z} \\ \text{or } (-\infty,\infty) = \mathbb{Z}\end{array}\right.$ is a geodesic*

if $\forall i$, σ_i is a simplex in $\mathcal{C}(S)$

and $\forall \alpha_i \in \sigma_i, \alpha_j \in \sigma_j$ $(i \neq j)$, $d_S(\alpha_i,\alpha_j) = |j-i|$

<u>Picture</u>:



<u>Note</u>: $\forall \alpha \in \sigma_{i-1}, \beta \in \sigma_{i+1}$, since $d_S(\alpha,\beta) = 2$, $i(\alpha,\beta) \neq 0$.

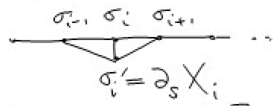
<u>Def</u>: $g = \{\sigma_i\}$ is <u>tight</u> at index j if $\sigma_j = \partial_S X(\sigma_{i-1}, \sigma_{i+1})$.

Def: g is tight if

1) g is a geodesic*
2) g is tight at every index
3) There are markings $I(g), T(g)$ [initial, terminal] s.t. $\sigma_0 \subseteq \text{base}(I(g))$

$$\sigma_N \subseteq \text{base}(T(g))$$

Lemma 1: Suppose g is geodesic* and g' is g, tightened at index i

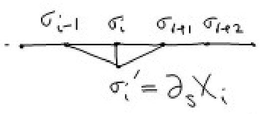
Picture: 

[define $X_i = X(\sigma_{i-1}, \sigma_{i+1})$]

Then g' is geodesic*.

Pf: Note that $\sigma_{i-1} \cup \sigma_i'$ and $\sigma_i' \cup \sigma_{i+1}$ are again simplices of $\mathcal{C}(S)$. Now apply triangle inequality. //

Lemma 2: Suppose g is tight at $i+1$. Then, all else as above, g' is tight at i and at $i+1$.

Pf: 

Let $X_{i+1}' = X(\sigma_i', \sigma_{i+2})$

[WTS: $\sigma_{i+1} = \partial_S X_{i+1}'$]

Step 1: $X_{i+1}' \subseteq X_{i+1}$.

Pf: Since g' is a geodesic* every component of σ_i' cuts σ_{i+2} $\Longrightarrow$ $\sigma_i' \cup \sigma_{i+2}$ is connected. Now $\sigma_{i+2} \subseteq X_{i+1}$ and $\sigma_i' \cap \partial_S X_{i+1}$

$$= \sigma_i' \cap \sigma_{i+1} = \emptyset.$$

$\Rightarrow \sigma_i' \subseteq X_{i+1} \Longrightarrow X_{i+1}' \subseteq X_{i+1}$ //(1)

Step 2: $X_{i+1}' = X_{i+1}$

Suffices to show that σ_i' and σ_{i+2} fill X_{i+1}.

Fix any $\gamma \in \mathcal{C}(X_{i+1})$ and suppose that $\gamma \cap \sigma_{i+2} = \emptyset$. [WTS: γ cuts σ_i']

a) γ cuts σ_i (b/c σ_i, σ_{i+2} fill X_{i+1})

b) γ cuts σ_{i-1} (b/c $\sigma_{i-1}, \sigma_{i+2}$ fill S)

c) Since $\sigma_{i-1} \subseteq X_i$ and $\sigma_i \cap X_i = \emptyset$ it follows that γ cuts $\partial_S X_i = \sigma_i'$ as desired. //2. //Pf that g' is tight at $i, i+1$.

Exercise: Tightening is not unique.

[Thm: $\forall \alpha, \beta$ $\exists$ only finitely many tight geod connecting them.]