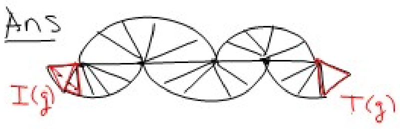
Lecture XII

<u>Question</u>: Why need $I(g)$, $T(g)$?

<u>Ans</u>



<u>Recall</u>; A <u>marking</u> μ is a <u>simplex</u> $\text{base}(\mu) \subseteq \mathcal{C}_0(S)$ and a collection $\text{trans}(\mu)$ of transversals $\tau_i \in \mathcal{C}(\alpha_i)$ for some $\alpha_i \in \text{base}(\mu)$.

<u>Thm</u> (Hierarchies exist)

<u>Def</u>: Say $H' = \{g\}$ is a <u>partial</u> hier if ① + ③ are satisfied and ②' is satisfied [replace "$\exists$ unique" by "$\exists$ at most one"]

If H' is a partial hierarchy and $Y \subseteq S$, $b, f \in H'$ s.t. $b \overset{d}{\swarrow} Y \overset{d}{\searrow} f$ and $\nexists\, k \in H'$ s.t. $D(k) = Y$, <u>then</u> call (Y, b, f) <u>unutilized</u>.

<u>Pf</u>, <u>Base case</u>: Given P, Q markings of S. Set $H_0 = \{g_H\}$ where $g = g_H$ is a tight geodesic connecting some $\alpha \in \text{base}(P)$ to some $\beta \in \text{base}(Q)$.
Let $I(g) = P$, $T(g) = Q$.

[<u>Question</u>: Have to choose α, β? Ans: Yes]
In fact [Exercise] tight geodesics generally not unique. [Exercise: Find α, β s.t $d_S(\alpha, \beta) \geq 3$ and $\exists$ unique tight geodesic between them.]

<u>Inductive Step</u>, Suppose H_n is partial and (Y_n, b_n, f_n) unutilized.

Let $M_k(n) = \#\left\{ Y \;\middle|\; \begin{array}{l} Y \text{ is a compt domain} \\ \text{of } H_n,\ k = \xi(Y) \end{array} \right\}$

$$v_n = \left(M_{-1}(n), M_1(n), M_2(n), \cdots, M_{\xi(S)}(n) \right)$$

Let $T(Y, f_n)$, $I(Y, b_n)$ be def. as before. pick $g_n \subseteq \mathcal{C}(Y_n)$ tight, with

$$I(g_n) = I(Y_n, b_n)$$
$$T(g_n) = T(Y_n, f_n)$$

Let $H_{n+1} = H_n \cup \{g_n\}$.

Notice that for all compt domains Z of g_n, $Z \subsetneq Y_n$ so $\xi(Z) < \xi(Y_n)$

<u>Thus</u> $v_{n+1} < v_n$ in lex-order. //

So take $H = \bigcup_{n=0}^{N} H_n$. //

Def: For H a hierarchy, $Y \subseteq S$

$$\Sigma^{+}_{H}(Y) = \left\{ f \in H \;\middle|\; \begin{array}{l} Y \subseteq D(f) \text{ and} \\ T(f)|_Y \neq \emptyset \end{array} \right\}$$

$\Sigma^{-}_{H}(Y)$ defined similarly

Thm [Structure of Σ]

① If $\Sigma^{\pm}_{H}(Y) \neq \emptyset$ then it has the form

$$\{ f_0 \overset{d}{\searrow} f_1 \overset{d}{\searrow} f_2 \overset{d}{\searrow} \cdots \overset{d}{\searrow} f_n = g_H \}$$

$$\{ g_H = b_m \overset{d}{\swarrow} b_{m-1} \overset{d}{\swarrow} \cdots \overset{d}{\swarrow} b_1 \overset{d}{\swarrow} b_0 \}$$

② If $\Sigma^{\pm}$ both nonempty then $f_0 = b_0$, all vertices of f_0 cut Y

③ If Y is a comp domain of some $k \in H$ Then $\forall f \in H$, $Y \searrow f$ iff $f \in \Sigma^{+}(Y)$

(Similarly for Σ^{-})

If both $\Sigma^{\pm} \neq \emptyset$ then $Y = D(f_0) = D(b_0)$

④ If $h, h' \in H$, $D(h) = D(h')$ then $h = h'$.

Lemma (Subord I) H hier, $Y \subseteq S$

1) if $Y \overset{d}{\searrow} h$ then $h \in \Sigma^{+}_{H}(Y)$

2) If $h \in \Sigma^{+}$, $h \searrow f$, then $f \in \Sigma^{+}$

3) If $Y \searrow f$ then $f \in \Sigma^{+}$.

Pf: (3) follows from ① + ② and def of $\searrow$. [Trans closure of $\overset{d}{\searrow}$].

(1) + (2) Next time.