Lecture XIII

Lemma (Sub. intersection 1)

① If $Y \overset{d}{\searrow} h$ then $h \in \Sigma^{+}_{H}(Y)$

② If $h \searrow f$, $h \in \Sigma^{+}$ then $f \in \Sigma^{+}$.

Pf: of ①: Since $Y \overset{d}{\searrow} h$, $\exists v_j \in h$ s.t.

Y is a component of $D(h) = v$ [Compt domain]

$\Rightarrow Y \subseteq D(h)$. Now, $T(Y, h) \neq \emptyset$ so

Case A: If v_j is last then

$T(Y,h) = T(h)|_Y \neq \emptyset$ so $h \in \Sigma^{+}$

Case B: If v_j is not last then

$T(Y,h) = v_{j+1}|_Y \neq \emptyset$. Since

$\emptyset_h(Y)$ is connected, $\forall w \in h$ if

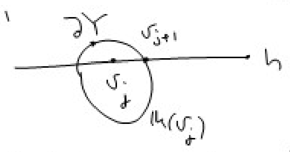
$v_{j+1} \leq w$ then w cuts Y.

Thus the last vertex of h also cuts

Y, so $T(h)$ cuts $Y \Rightarrow T(h)|_Y \neq \emptyset$.

$\Rightarrow h \in \Sigma^{+}$.

Picture:



Pf of 2: Induct on m where $h = f_0 \overset{d}{\searrow} f_1 \overset{d}{\searrow} \cdots \overset{d}{\searrow} f_m = f$

If $m = 0$ this is vacuous. Induction step reduces to case $m = 1$ ..

So suppose $h \in \Sigma^{+}$, $h \overset{d}{\searrow} f$, WTS $f \in \Sigma^{+}$.

So: $D(h)$ is a compt domain of $v_j \in f$

and $T(D(h), f) \neq \emptyset$. Note $Y \subseteq D(h) \subseteq D(f)$.

Case A: $v_j \in f$ is last.

So $T(h) = T(D(h), f) = T(f)|_{D(h)} \neq \emptyset$

Since $h \in \Sigma^{+}(Y)$ we are told $T(h)|_Y \neq \emptyset$

Hence $T(f)|_Y \neq \emptyset$ as desired.

$\Rightarrow f \in \Sigma^{+}$.

Case B: $v_j \in f$ is not last.

So $T(h) = v_{j+1}|_{D(h)} \neq \emptyset$

Again $T(h)|_Y \neq \emptyset$ [b/c $h \in \Sigma^{+}$]

Since footprints are connected

max f cuts Y and so does $T(f)$. //

Corollary: (Footprints) $h, f \in H$, $h \in \Sigma^{+}$

$h \searrow f$ then $\max \emptyset_f(h) = \max \emptyset_f(Y)$.

Pf: Exercise [If $Z \subseteq W$ then

$\emptyset_g(W) \subseteq \emptyset_g(Z)$] //

Lemma: (Uniqueness of Descent)

H a hierarchy, $Y \subseteq S$

1) If $\Sigma^{+} \neq \emptyset$ then

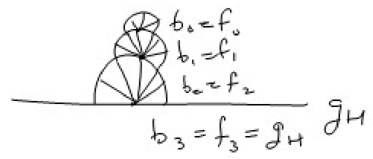
$$\Sigma^{+} = \{ f_0 \overset{d}{\searrow} f_1 \overset{d}{\searrow} f_2 \overset{d}{\searrow} \cdots \overset{d}{\searrow} f_n = g_H \}$$

[and similarly for Σ^{-}]

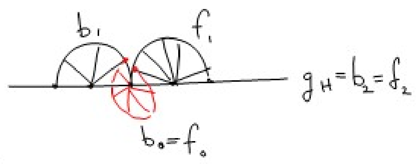
2) if $\exists h \in H$ $D(h) = Y$ then $h = f_0 = b_0$

Σ^{+}? $\quad$ Σ^{-}

Example:



Another example



Pf (Descent): ② follows from ①.

Pf of (1) Ascend inductively [Induct on $\xi(S) - \xi(Y)$.] Base case, If $Y = S$ then $\Sigma_1^{\pm} = \{g_H\}$. ✓

Inductive step: Suppose $f, f', g \in \Sigma_1^+$ with $f \overset{d}{\searrow} g$, $f' \overset{d}{\searrow} g$. So $D(f), D(f')$ are compt domains of max $\phi_g(f)$, max $\phi_g(f')$. By corollary we know

$$\max \phi_g(f) = \max \phi_g(Y) = \max \phi_g(f')$$

Since $Y \subseteq D(f)$ and $Y \subseteq D(f')$, $D(f) = D(f')$.

If $\xi(D(f)) > \xi(Y)$ then by ② [2nd conclus. of Lemma] [ind. hypothesis] f is the unique geod in Σ_1^+ s.t. $f \overset{d}{\searrow} g$. All this also holds for Σ_1^-. Finally suppose h, h' have $D(h) = Y = D(h')$. So, arguing as above $\exists\, f_1, b_1$ s.t.

$$b_1 \overset{d}{\swarrow} h \overset{d}{\searrow} f_1 \quad \text{and}$$
$$b_1 \overset{d}{\swarrow} h' \overset{d}{\searrow} f_1 .$$

So by the uniqueness in part ② [def of hierarchy] $h = h'$, as desired. //

Lemma: $k, h \in H$, $h \in \Sigma_1^+(D(k)) \Rightarrow k \overset{\searrow}{=} h$

Cor: If $D(k) \subseteq D(h) \Rightarrow \phi_h(k) \neq \emptyset$

Lemma: $Y \overset{d}{\searrow} h$, $\Sigma_1^+ = \{f_0, f_1, \dots, f_N = g_H\}$

$\Sigma_1^- = \{g_H = b_M, b_{M-1}, \cdots b_0\}$

either $h = f_1$, $Y = D(f_0)$ or $h = f_0$ and $Y \neq D(k)\ \forall k \in H$.

Time order $h, h' \in H$.

Say $h <_t h'$ [precedes in time order]

if $\exists\, m \in H$ s.t. $\max \phi_m(h) < \min \phi_m(h')$

[We call m the comparison geodesic.]

WTS: $<_t$ is a partial order on H.