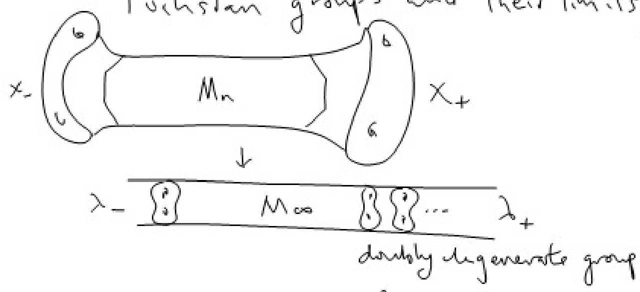
Lecture XV:

Applications of Hierarchies

① Distance estimates for $\mathcal{M}(S)$, $P(S)$

② Linear Bounds on lengths of conjugating words in $MCG(S)$.
[Fix $\underline{X} \subset MCG(S)$ a gen set and suppose $f, g \in MCG(S)$ are conjugate: then $\exists\, w \in MCG(S)$ s.t. $wg\bar{w} = h$ and $|w| \leq C(|g| + |h|)$]

③ ELC ending lamination conjecture
[more generally, the structure of quasi-Fuchsian groups and their limits]



④ The geometry of the disk complex of a handlebody [Relevant to the study of $MCG(V_g)$; V a handlebody]

⑤ Inspires questions about

$GL(n,\mathbb{Z})$	{Abelian Varieties}	Building
$MCG(S)$	$Teich(S)$	$\mathcal{C}(S)$
$Out(F_n)$	X_n	?

More on ④

$D(V)$ = the complex of essential disks.

Note $MCG(V) \hookrightarrow MCG(\partial V)$

$$\downarrow$$

$$Out(F_n) = Out(\pi_1(V))$$

<u>Def</u>: Say H is <u>complete</u> if $\forall\, Y$ compt domain of $h \in H$ $\exists\, k \in H$ s.t. $Y = D(k)$.

<u>Thm</u>: If $I(H)$, $T(H)$ are complete, so is H.

<u>Pf</u>: $\forall\, Y$ comp domain of some $h \in H$,
Note that $I(H)|_Y \neq \emptyset$ and $T(H)|_Y \neq \emptyset$
Thus $g_H \in \Sigma_1^{\pm}(Y)$, so $g_H \swarrow Y \searrow g_H$
$\Rightarrow$ Y is the support of some geodesic by property ② in def. //

<u>Slices</u>: Say $\tau = \{(h, v)\}$ is a <u>slice</u> of H if

(1) $\forall\, (h,v) \in \tau$, $v \in h \in H$.

(2) $\forall\, h \in H$, h appears in at most 1 pair in τ

(3) $\exists!$ <u>bottom pair</u> $(h_\tau, v_\tau) \in \tau$

(4) If $(k, w) \in \tau$ [not bottom pair] then $\exists\, (h, v) \in \tau$ s.t. $D(k)$ is a compt of $(D(h), v)$

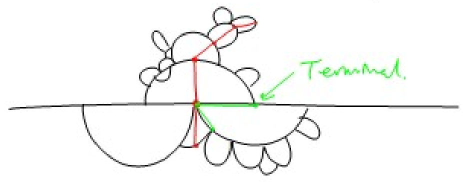
Exercise: τ is finite.

If additionally we have

(5) $\forall (h,v) \in \tau$ $\forall$ Y comps of (h,v)

$\exists (k,w) \in \tau$ s.t. $D(k) = Y$.

Then we call τ complete.



Definition: Say τ is $\left\{ \begin{matrix} \text{initial} \\ \text{terminal} \end{matrix} \right\}$

if $\forall (h,v) \in \tau \setminus (h_\tau, v_\tau)$, v is $\left\{ \begin{matrix} \text{first} \\ \text{last} \end{matrix} \right\}$ in h.

Slices to Markings

Induction shows that

$$\bigcup_{\substack{(h,v)\in\tau \\ D(h) \neq A^2}} v = \text{base}(\mu_\tau) \text{ is a multicurve.}$$

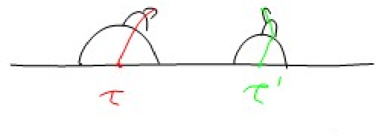
Let $\text{trans}(\mu_\tau) = \{ (D(h), v) \}_{\substack{D(h) = A^2 \\ (h,v) \in \tau}}$

Def Suppose H complete. Define $V(H) =$

$$\left\{ \tau \;\middle|\; \begin{matrix} \text{complete slice of } H \\ \text{with bottom pair on } g_H \end{matrix} \right\}$$

$\forall \tau, \tau' \in V(H)$ write $\tau <_s \tau'$ iff $\tau \neq \tau'$ and $\forall (h,v) \in \tau$ either $(h,v) \in \tau'$ or $\exists (h',v') \in \tau'$ s.t.

$$(h,v) <_p (h',v')$$



Lemma: $<_s$ is a strict partial order.

Pf: Suppose $\tau_1 <_s \tau_2 <_s \tau_3$

So $\forall p_i \in \tau_i$ $\exists p_{i+1} \in \tau_{i+1}$ s.t. either $p_i = p_{i+1}$ or $p_i <_p p_{i+1}$. Since $\tau_1 \neq \tau_2$ $\exists$ $p_1 <_p p_2$. Thus $p_1 <_p p_2 \leq_p p_3$ $\Rightarrow p_1 <_p p_3$ so $\tau_1 <_s \tau_3$. //

Exercise: Build a Hierarchy in $S_{0,5}$ and find some slices of it.