Lecture XVII

Suppose H is a hierarchy 

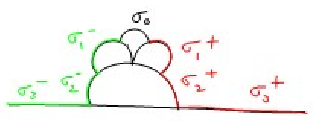
Suppose Y is a domain.

Define: $\sigma_i^+(Y) =$

$$\left\{ (f_i, v_j) \;\middle|\; \begin{array}{l} f_i \text{ ith elt of } \Sigma^+(Y) \\ \max \phi_{f_i}(Y) < v_j \end{array} \right\}$$

$\sigma_j^-(Y)$ similarly

$\sigma_0 = h$ where $D(h) \supseteq Y$, $h = b_0 = f_0$.

Picture: 

Let $\sigma^+ = \sum \sigma_i^+$, $\sigma^- = \sum \sigma_j^-$

$$\sigma = \sigma^- \cup \sigma_0 \cup \sigma^+$$

Lemma (Sigma projection)
$\exists M_1, M_2 \geq 0$ depending on S only

① if $Y = D(h)$ then

$$\operatorname{diam}_Y(\pi_Y(\sigma^+)) \leq M_1$$

and sim. for σ^-

② if $Y \subsetneq D(h)$

$$\operatorname{diam}_Y(\pi_Y(\sigma)) \leq M_2.$$

Pf: ② $\forall (k,v) \in \sigma$ $v|Y \neq \emptyset$ so we may apply Bounded geodesic image theorem: to the data $Y, \sigma_i^+, D(f_i)$ [or $Y, \sigma_j^-, D(b_j)$]

So $\operatorname{diam}_Y(\pi_Y(\sigma_i^+))$ bounded. [or $Y, h, D(h)$]

Also the terminal marking of f_i is contained in some $v \in \sigma_{i+1}^+$.

Since $|\Sigma^+(Y)| \leq \xi(S)$ done. //

① is identical except we omit h as $h \subseteq \zeta(Y)$ so BGIT does not apply. //

Lemma (Largelink) $Y \subseteq S$, H given

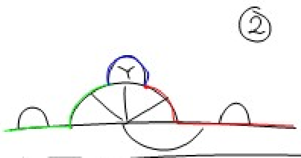
① If $d_Y(I(H), T(H)) > M_2$ then $\exists h \in H$ s.t. $D(h) = Y$

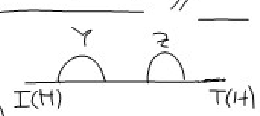
② If $h \in H$, $Y = D(h)$ then

$$\big|\, |h| - d_Y(I, T) \,\big| \leq 2M_1$$

Pf: Exercise, contrapos of ① [use ② Sigma Proj]

② use ① sigma proj. //





Lemma (Order+Projection)

Suppose $h, k \in H$, $Y, Z = D(h), D(k)$

Y, Z overlap, suppose $h <_t k$ Then

① $d_Y(\partial Z, T(H)),\ d_Z(I(H), \partial Y) \leq M_1 + 2$

Let m be the comparison geodesic.

Pick $v \in \emptyset_m(Z)$, $(m, v) \in \sigma^+(Y)$

$\Rightarrow d_Y(v, T(H)) \leq M_1$ $\quad \left[(g_H, T(H)) \pitchfork \sigma^+(Y) \right]$

<u>But</u> v misses Z

so $d_Y(v, \partial Z) \leq 2$. //

<u>Recall</u>: $\exists B \geq 0$ s.t. $d_{\mathcal{M}}(I, T) \leq B \cdot |H|$

To prove a lower bound is <u>harder</u>.

Let $M_5 = 2M_1 + 5$, $M_6 = 4(M_1 + M_5 + 4)$

$\forall M \geq M_6$ define $G = G_M = \{h \in H \mid |h| \geq M\} = G_M(H)$.

Define $|G| = \sum'_{h \in G} |h|$

[Good <u>Exercise</u>: $\forall M \geq 0$ $\exists A \geq 1$, $\forall$ hier. H.

$$|H| \leq_A |G| \leq |H|$$

<u>Thm</u>: $\forall S$ $\forall M \geq M_6 = M_6(S)$ $\exists A$

$\forall$ complete H

$$d_{\mathcal{M}}(I(H), T(H)) =_A |G_M(H)|$$

Ignore for now. $\left[\quad =_{A'} \sum'_{Y \subseteq S} [d_Y(I, T)]_{M'} \right]$

<u>Pf</u> of the bound $|G| \leq_A d_{\mathcal{M}}(I, T)$

Let $\mu_0 = I(H)$. Let $N = d_{\mathcal{M}}(I, T)$.

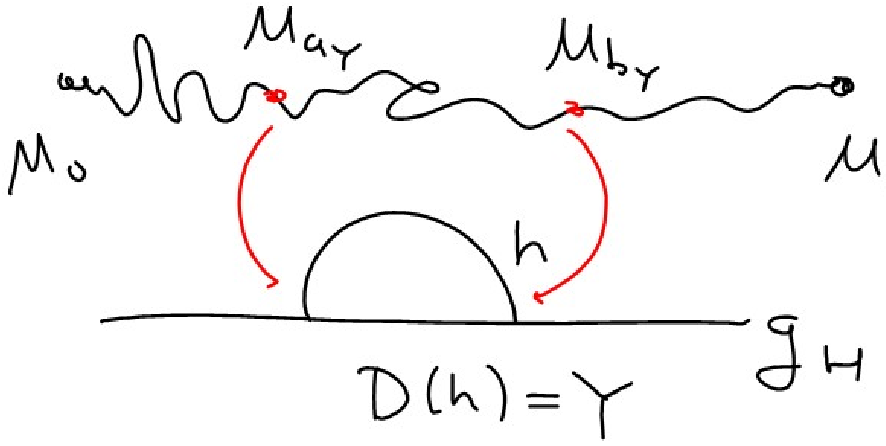
Let $\mu_N = T(H)$. $\{\mu_i\}_{i=0}^{N}$ a path in $\mathcal{M}(S)$

Fix any $Y \subseteq S$. Note $d_Y(\mu_i, \mu_{i+1}) \leq 4$

Suppose $Y = D(h)$, $h \in G_M$. Define:

$$a_Y = \max\left\{ j \in [0, N] \;\middle|\; d_Y(\mu_0, \mu_j) \in [M_S, M_S + 4] \right\}$$

$$b_Y = \min\left\{ j \in [a_Y, N] \;\middle|\; d_Y(\mu_j, \mu_N) \in [M_S, M_S + 4] \right\}$$



Define $J_Y = [a_Y, b_Y]$

<u>Exercise</u>: a_Y, b_Y well defined.

<u>Idea</u>: Inside of J_Y, "μ_j moves through Y"