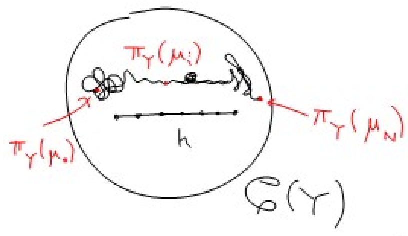
Lecture XVIII:

Given $\{\mu_i\}_{i=0}^{N}$ a path in $\mathcal{M}(S)$ connecting $I(H), T(H)$. [WTS $|H| \leq_A N$] Recall $G_M(H) = G =$
$\{h \in H \mid |h| \geq M\}$

Suffices to show $|G| \leq_A N$.

Last time: Defined $J_Y = [a_Y, b_Y] \subseteq [0, N]$



Def:
$a_Y = \max\{ j \in [0,N]$ s.t. $d_Y(\mu_0, \mu_j) \in [M_5, M_5+4]\}$

$b_Y = \min\{ j \in [a_Y, N] \mid d_Y(\mu_j, \mu_N) \in [M_5, M_5+4]\}$

Properties of the J_Y

① $\forall j \in J_Y \quad d_Y(\mu_0, \mu_j),\ d_Y(\mu_j, \mu_N) \geq M_5$

② $|J_Y| \geq |h|/8 \quad [D(h) = Y]$

③ If $h, k \in G$, $Y, Z = D(h), D(k)$ s.t. Y, Z overlap, then $J_Y \cap J_Z = \emptyset$

Pf: ① follows from def of J_Y and Lemma $d_Y(\mu_j, \mu_{j+1}) \leq 4$. //①

② $|J_Y| = b_Y - a_Y \geq \frac{1}{4} d_Y(\mu_a, \mu_b)$

Let $L = d_Y(\mu_0, \mu_N)$

By large link $L \geq |h| - 2M_1$

By Δ inequality $d_Y(\mu_0, \mu_b) \geq L - M_5 - 4$

By def $d_Y(\mu_0, \mu_a) \leq M_5 + 4$

So: $L - M_5 - 4 \leq d_Y(\mu_0, \mu_b) \leq d_Y(\mu_0, \mu_a) + d_Y(\mu_a, \mu_b)$

Deduce $d_Y(\mu_a, \mu_b) \geq L - 2(M_5+4)$

$\geq |h| - 2(M_5 + 4 + M_1)$

[Since $|h| > M \geq M_6 = 4(M_1 + M_5 + 4)$]

$\geq |h|/2$

$\Rightarrow |J_Y| \geq |h|/8$. //②

③ Y, Z overlap. Suppose $j \in J_Y \cap J_Z$ [for contradiction]. Let $\mu = \mu_0$, $\mu' = \mu_j$

Let H' be a hierarchy between μ and μ'.

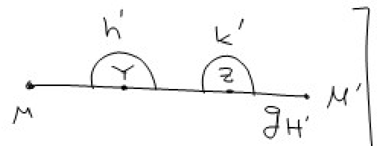
Since $d_Y(\mu, \mu'), d_Z(\mu, \mu') > M_5$ $\quad$ [$M_5 > M_2 = M_1$ let us say]

we find $h', k' \in H'$ with $D(h'), D(k') = Y, Z$

[Follows from large link]

Since Y, Z overlap, h', k' are time-ordered. WLOG $h' <_t k'$.

By order and projections



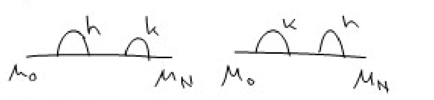
$d_Y(\partial Z, \mu') \le M_1 + 2$

and

$d_Z(\mu, \partial Y) \le M_1 + 2$

Also $d_Y(\partial Z, \mu_N) \le M_1 + 2$ or

$d_Y(\mu_0, \partial Z) \le M_1 + 2$ [as we have $h <_t k$ or $k <_t h$]



So either $d_Y(\mu', \mu_N) \le 2M_1 + 4$ or

$d_Y(\mu', \mu_0) \le 2M_1 + 4$

But: $\mu' = \mu_j$, $j \in J_Y$. Since $M_5 > 2M_1 + 4$

this contradicts def of b_Y or a_Y. // ③

[Exercise $\forall j \in [0, N]$ the set $\{ Y \subseteq S \mid j \in J_Y \}$ has card $\le 2\xi(S)$]

So $2\xi(S) \cdot N \ge \sum'_{\substack{Y = D(h) \\ h \in G}} |J_Y| \ge \frac{|G|}{8}$

Since $|G| \ge_A |H|$, we are done. // Thm

Restate: [Exercise]

$\forall S \; \exists M_6 = M_6(S) \; \forall M \ge M_6 \; \exists A \ge 1$
$\forall \mu, \nu \in \mathcal{M}(S)$

$$d_{\mathcal{M}}(\mu, \nu) =_A \sum'_{Y \subseteq S} [d_Y(\mu, \nu)]_M$$

[Hint: Check that when $|h| = M$, $d_Y < M$, everything still ok]

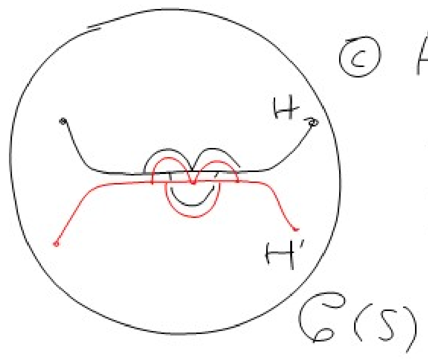
Other topics :

① Finiteness properties of hierarchies:

ⓐ $\exists$ only fin. many hierarchies connecting μ and $\nu \in \mathcal{M}(S)$

ⓑ If H, H' are hierarchies with

$$d_m(I(H), I(H')), d_m(T(H), T(H')) \leq K$$



ⓒ Another application of similar ideas: If $f \in MCG(S)$ is pseudo-Anosov then $\exists N$ and a hierarchy H so that f^N leaves H invariant. [Bowditch]

② Pants graph has vertex set $\left\{\begin{matrix}\text{pants}\\ \text{decomps}\end{matrix}\right\}$

edges for $\{\text{flips}\}$

Thm [MM]

$\forall S\ \exists M_0\ \forall M \geq M_0\ \exists A \geq 1\ \forall P, Q \in \mathcal{P}(S)$

$$d_{\mathcal{P}}(P, Q) =_A \sum_{\substack{Y \subseteq S \\ Y \not\cong A^2}} \left[d_Y(P, Q) \right]_M$$

$\exists$ theory of distance estimates for combinatorial complexes [Arc complex, Hatcher-Thurston, Nonsep, Disk complex, ..., Symmetric curve complex...]