MA 3F20

THE UNIVERSITY OF WARWICK

THIRD YEAR EXAMINATION: JUNE 2009

KNOT THEORY

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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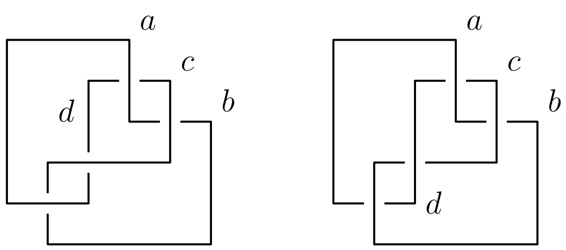
1. a) Define the notion of a *diagram* of a link. **[3]**

b) Define the Reidemeister moves on a diagram. **[3]**

c) Define the *coloring group*, Col(D), of a diagram D. **[3]**

d) Compute the coloring groups of the following diagrams. **[6]**



e) Suppose that D is a diagram of the link K. Show that Col(D) is an isotopy invariant of K. Clearly state any theorems you use from class. **[10]**

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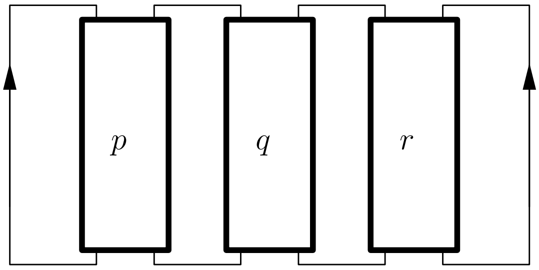
2. a) Define *twist boxes.* **[2]**

b) Suppose that K is an oriented link. Define ∇_K, the Conway polynomial of K. **[3]**

c) Define, using a twist box diagram, the *twist knots* T_k. **[3]**

d) Let $P_k(z) = \nabla_{T_k}(z)$. Compute P_k for $k \geq -1$. Clearly state any theorems you use from class. **[10]**

e) Convert the twist box diagram given below into a standard diagram, using the values $p = 3, q = 2, r = 3$. Compute ∇_K. Clearly state any theorems you use from class. **[7]**



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3. a) Define the braid group B_n by giving generators and relations. **[3]**

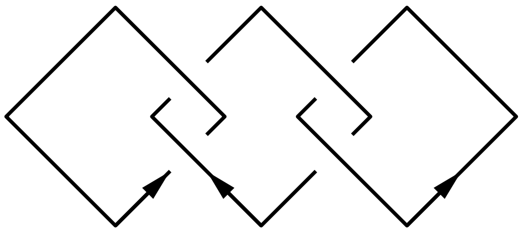
b) Define the Markov moves on braids. **[3]**

c) Define the *closure* of a braid. **[3]**

d) State (without proof) Markov's Theorem. **[3]**

e) Draw and identify the closures of the braids $(\sigma_1\sigma_3)^3$, $(\sigma_1\sigma_2^{-1})^2$, and $(\sigma_1\sigma_2^{-1})^3$. **[6]**

f) Convert the diagram below to be a braid closure, using the algorithm given in class. Record the resulting braid word. **[7]**



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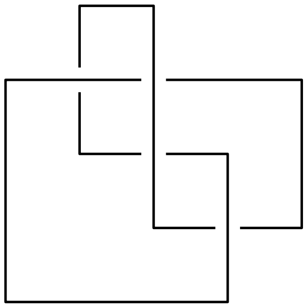
4. a) Give the axioms for the *Kauffman bracket* in terms of the variables A, B, C. **[3]**

b) Define the *Kauffman polynomial.* **[3]**

c) Define *Kauffman states* and the Kauffman *state sum* formulation of the Kauffman bracket. **[6]**



d) Produce the A/B labelling of the diagram shown. List all of the Kauffman states of the diagram. **[6]**

e) Using the state sum formulation prove that $\langle D \cup E \rangle = C \cdot \langle D \rangle \langle E \rangle$. (Here $D \cup E$ is the disjoint union of the diagrams D and E.) **[7]**

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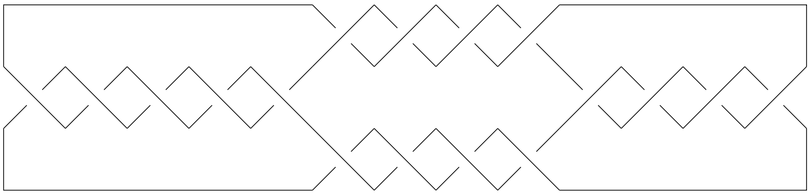
5. a) Define *twist boxes*. **[2]**

b) Define a *tangle*. **[3]**

c) Define a *flype*. **[4]**

d) Give a careful proof that the knot given below is isotopic to the unknot. **[7]**



e) Define the *numerator* $N(T)$ of a tangle T. **[3]**

f) Draw the rational tangle 23/10. Draw $N(T)$. Compute $\det(N(T))$. **[6]**

MA 3F20

MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS

Course Title: KNOT THEORY

Model Solution No: 1

a) Define the notion of a *diagram* of a link. **[3]**

A diagram is a four-valent graph in $\mathbb{R}^2$ with finitely many vertices (*crossings*) and smooth edges (*arcs*). At each crossing the first and third (second and fourth) edges form a smooth path. At each crossing there is crossing information – the path formed by the first and third edges is either either passes over or under the path formed by the other two edges. A diagram D is the *projection* of a link L if D is obtained by a orthogonal projection of $L \subset \mathbb{R}^3$ into the xy–plane.

b) Define the Reidemeister moves on a diagram. **[3]**

There are four Reidemeister moves. R_0 is planar isotopy. Each of R_1, R_2, R_3 is best shown by a picture. The R_1 move adds a twirl to an arc. The R_2 move takes two arcs and passes a segment of one under the other. The R_3 move takes three arcs and passes a segment of the third under a crossing of the first two. (The R_0 move is not required for full marks.)

c) Define the *coloring group*, $\mathrm{Col}(D)$, of a diagram D. **[3]**

This is an Abelian group. There is one generator for each overcrossing arc. There is one relation $a + b = c$ for every crossing, where a, b are the undercrossing arcs and c is the overcrossing. Also, one arc is picked and the associated generator (say a) is set to zero ($a = 0$).

d) Compute the coloring groups of the following diagrams.

The first diagram is of the trefoil. The coloring group is:

$$\langle a, b, c, d \mid a, a + b - 2c, b + c - 2a, c + d - 2a, d + a - 2c \rangle.$$

Since $a = 0$ this simplifies to:

$$\langle b, c, d \mid b - 2c, b + c, c + d, d - 2c \rangle.$$

Since $b = d = -c$ this becomes:

$$\langle c \mid -3c \rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

[3]

The second diagram is of the figure eight knot. The coloring group is:

$$\langle a, b, c, d \mid a, a + b - 2c, b + c - 2d, c + d - 2a, d + a - 2b \rangle.$$

Since $a = 0$ the group is isomorphic to:

$$\langle b, c, d \mid b - 2c, b + c - 2d, c + d, d - 2b \rangle.$$

Since $d = -c$ the group is isomorphic to:

$$\langle b, c \mid b - 2c, b + 3c, c + 2b \rangle.$$

Since $b = 2c$ we have:

$$\langle c \mid 5c \rangle \cong \mathbb{Z}/5\mathbb{Z}.$$

[3]

e) Suppose that D is a diagram of the link K. Show that $\mathrm{Col}(D)$ is an isotopy invariant of K. Clearly state any theorems you use from class.

By a theorem in class, the coloring group is the cokernel of A, the reduction of the matrix of coloring equations. By an exercise from the example sheets the Smith normal form, and hence the cokernel, of A is independent of the column deleted from A_+. Thus the choice of special relation $a = 0$ does not effect the coloring group. **[3]**

We now show that the coloring group is invariant under the Reidemeister moves. By Reidemeister's theorem it will then follow that $\mathrm{Col}(D)$ is a link invariant. **[2]**

An R_1 move on the arc a introduces a new generator b and a relation of the form $a + b = 2b$ or $a + b = 2a$, depending on the handedness of the move. Thus we have the relation $a = b$ and the group is unchanged. **[1]**

An R_2 move between the arcs a, b introduces two new generators c, d and, without loss of generality, two relations $b + c = 2a$ and $c + d = 2a$. Thus we deduce that $b = d$ and c may be defined in terms of the other generators. It follows that the group is unchanged. **[1]**

An R_3 move among the generators a (overcrossing), b, c (second layer), and d, e, f (undercrossing) transforms these into generators a, b, c, d, e', f. The relations

$$b + c = 2a, \quad d + e = 2c, \quad e + f = 2a$$

transform into

$$b + c = 2a, \quad d + e' = 2a, \quad e' + f = 2b.$$

Thus e and e' may be eliminated from the each generating sets giving relations $d - f = 2c - 2a$ and $d - f = 2a - 2b$ respectively. In the presence of the relation $b + c = 2a$ these both reduce to $d - f = c - b$ and we are done. **[3]**

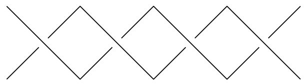
MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS

Course Title: KNOT THEORY

Model Solution No: 2

a) Define *twist boxes*. **[2]**

Here is a picture of a twist box of size four:



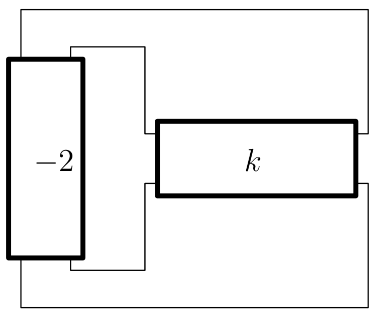
In general the twist box of size p contains p crossings connected as shown above. The signs of the crossings (measured along the axis of the box) agree with the sign of p. When p is zero the twist box contains two parallel strands.

b) Suppose that K is an oriented link. Define ∇_K, the Conway polynomial of K. **[3]**

The Conway polynomial has three axioms:

- ∇_K is a link invariant.
- $\nabla_U = 1$, where U is the unknot.
- Suppose that D is a diagram c is a crossing of D. The Conway polynomial satisfies a skein relation:
$$\nabla_+ - \nabla_- = z\nabla_0.$$
This relates the Conway polynomials of D with a positive crossing at c, of D with a negative crossing at c, and of D with the crossing c smoothed away.

c) Define, using a picture with twist boxes, the *twist knots* T_k. **[3]**



d) Let $P_k(z) = \nabla_{T_k}(z)$. Compute P_k for $k \geq -1$. Clearly state any theorems you use from class. **[10]**

There are two cases, depending on the parity of k. Suppose that $k = 2n$ is even. When $n = 0$ the knot T_0 is the unknot and so $P_0 = 1$. Now suppose that $2n > 0$. Applying the skein relation to a crossing in the twist box gives a recurrence relation:

$$P_{2n-2} - P_{2n} = z\nabla_H$$

where H is the Hopf link with linking number one. Thus $\nabla_H = z$ and induction on the recurrence relation proves that $P_{2n} = 1 - nz^2$.

Now suppose that $k = 2n + 1$ is odd. When $n = -1$ the knot T_{-1} is the unknot and so $P_{-1} = 1$. Now suppose that $2n + 1 > 0$. Applying the skein relation to a crossing in the twist box gives a recurrence relation:

$$P_{2n-1} - P_{2n+1} = z\nabla_{H'}$$

where H' is the Hopf link with linking number minus one. Thus $\nabla_{H'} = -z$ and induction on the recurrence relation proves that $P_{2n-1} = 1 + nz^2$.

(Students may also use, without proof, the fact that $\nabla_T = 1 + z^2$ where T is the left or right trefoil knot. This replaces the basis case for odd parity.)

e) Draw a diagram for the knot K given below, for $p = 3, q = 2, r = 3$. Compute ∇_K. Clearly state any theorems you use from class. **[7]**

The resulting link L is a clasp sum of a pair of right handed trefoils. Let c be a crossing in the twist box of size two. With the given orientations c is a positive crossing. Switching the sign of c gives a splittable link L'. Smoothing at c gives $T \# T$, the connect sum of trefoils. By a theorem from class $\nabla_{L'} = 0$ for any splittable link. By a theorem from class $\nabla_{M\#N} = \nabla_M \cdot \nabla_N$. The skein relation gives:

$$\nabla_L - \nabla_{L'} = z\nabla_{T\#T}$$

and so

$$\nabla_L = z(1 + z^2)^2 = z + 2z^3 + z^5.$$

MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS

Course Title: KNOT THEORY

Model Solution No: 3

a) Define the braid group B_n by giving generators and relations. **[3]**

Fix $n > 1$. The braid group B_n on n strands is generated by $n-1$ elements $\{\sigma_i\}$. There are two kinds of relations: far commutivity

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

when $|i-j| > 1$ and the braid relation

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$$

when $|i-j| = 1$.

b) Define the Markov moves on braids. **[3]**

Suppose that $\sigma \in B_n$ is a braid. There are two Markov moves. The M_1 move is conjugation: σ is replaced by $\tau\sigma\tau^{-1}$ for some $\tau \in B_n$. The M_2 move is *stabilization*: σ is replaced by $\sigma\sigma_n^{\pm 1} \in B_{n+1}$.

c) Define the *closure* of a braid. **[3]**

If σ is a braid then the link L_σ is the *closure*: we connect the points $(i, 1)$ to $(i, 0)$ in the xy–plane without introducing new crossings. (Full marks will be given for giving the correct picture.)

d) State (without proof) Markov's Theorem. **[3]**

Braid closures L_σ and L_τ are isotopic if and only if the braids σ and τ are related by a sequence of Markov moves.

e) Draw and identify the closures of the braids $(\sigma_1\sigma_3)^3$, $(\sigma_1\sigma_2^{-1})^2$, and $(\sigma_1\sigma_2^{-1})^3$. **[6]**

These are a disjoint union of trefoil knots, the figure eight knot, and the Borromean rings, respectively.

f) Convert the diagram into a braid closure, using the algorithm given in class. Record the resulting braid word. **[7]**

Upon oriented smoothing of all crossings there are three Seifert circles and these are not nested. There is exactly one disagreement and hence exactly one conflict. (In fact this appears between the outer two circles.) After resolving the conflict there are no disagreements. There is a pair of nested circles. An R_∞ move gives three nested circles and an R_0 move straightens the resulting braid closure. The braid word is $\sigma_1^{-1}\sigma_2^{-2}\sigma_1\sigma_2^{-2}$.

MA 3F20

MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS

Course Title: KNOT THEORY

Model Solution No: 4

a) Give the axioms for the *Kauffman bracket* in terms of the variables A, B, C. **[3]**

Suppose that D is a diagram of K and c is a crossing of D. The Kauffman bracket satisfies the following axioms:

(i) $\langle U \rangle = 1$ where U is the diagram of the unknot with no crossings.

(ii) $\langle D \cup U \rangle = C\langle D \rangle$ where $DcupU$ is the disjoint union of D and U.

(iii) $\langle D \rangle = A\langle D_R \rangle \langle D_L \rangle$ where D_R and D_L are the right and left smoothings of D at c.

b) Define the *Kauffman polynomial*. **[3]**

Let $w = w(D)$ be the writhe of D. Then the Kauffman polynomial of K is defined as

$$X_K(A) = (-A^{-3})^w \langle D \rangle$$

where we take $B = A^{-1}$ and $C = -A^2 - A^{-2}$.

c) Define *Kauffman states* and the Kauffman *state sum* formulation of the Kauffman bracket. **[6]**

A Kauffman state s is a function from the set of crossings of D to the set $\{\text{left}, \text{right}\}$. A state s defines a complete smoothing of D. We define $a(s)$ to be the number of right smoothings, $b(s)$ to be the number of left smoothings, and $c(s)$ to be the number of circles that result.

It follows that the Kauffman bracket is realized as

$$\langle D \rangle = \sum A^{a(s)} B^{b(s)} C^{c(s)-1}$$

where the sum ranges over states of D.

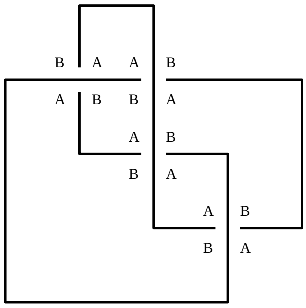
d) Produce the A/B labelling of the diagram shown. List all of the Kauffman states of the diagram. **[6]**

There are $2^4 = 16$ states. We simply list them in a canonical order, given an ordering of the crossings.

$$AAAA, AAABC, AABAC, AABBC^2, ABAAC, ABAB, ABBA, ABBBC,$$

$$BAAAC, BAAB, BABA, BABBC, BBAAC^2, BBABC, BBBAC, BBBBC^2$$

(Students may provide diagrams. Marks will be given only if some organizational material is provided.)



e) Using the state sum formulation prove that $\langle D \cup E \rangle = C \cdot \langle D \rangle \langle E \rangle$. (Here $D \cup E$ is the disjoint union of the diagrams D and E.) **[7]**

A state s for $D \cup E$ determines states s_D, s_E for D, E. The invariants add: $a(s) = a(s_D) + a(s_E)$, $b(s) = b(s_D) + b(s_E)$, and $c(s) = c(s_D) + c(s_E)$. Thus we find

$$\begin{aligned}
\langle D \cup E \rangle &= \sum A^{a(s)} B^{b(s)} C^{c(s)-1} \\
&= \sum A^{a(s_D)+a(s_E)} B^{b(s_D)+b(s_E)} C^{c(s_D)+c(s_E)-1} \\
&= C \cdot \left(\sum A^{a(s_D)} B^{b(s_D)} C^{c(s_D)-1} \right) \left(\sum A^{a(s_E)} B^{b(s_E)} C^{c(s_E)-1} \right) \\
&= C \cdot \langle D \rangle \langle E \rangle.
\end{aligned}$$

Here the first two sums are over states of $D \cup E$ and the latter two are over states for D and E respectively.

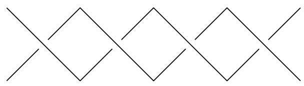
MATHEMATICS DEPARTMENT
THIRD YEAR UNDERGRADUATE EXAMS

Course Title: KNOT THEORY

Model Solution No: 5

a) Define *twist boxes*. **[2]**

Here is a picture of a twist box of size four:

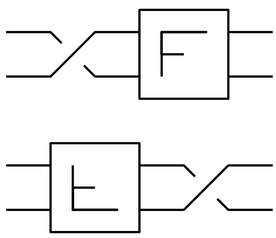


In general the twist box of size p contains p crossings connected as shown above. The signs of the crossings (measured along the axis of the box) agree with the sign of p. When p is zero the twist box contains two parallel strands.

b) Define a *tangle*. **[3]**

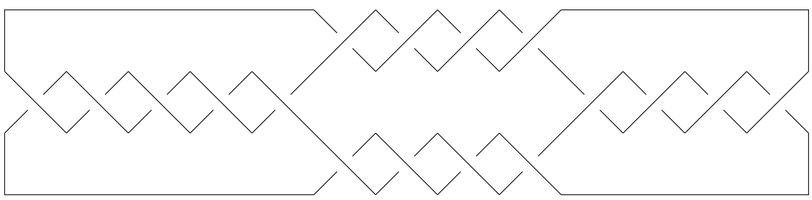
Let $\mathbb{B}$ be a three-ball. A *tangle* is a collection of circles and exactly two arcs embedded in $\mathbb{B}$, having intersection with $\partial\mathbb{B}$ being exactly the endpoints of the two arcs. Notice that a twist box is a special case of a tangle.

c) Define a *flype*. **[4]**



Suppose that F is a tangle, and it is added to a crossing as shown in the first line of the figure above. Then a 180 degree rotation of F in space causes F to commute with the crossing, as shown on the second line.

d) Give a careful proof that the knot given below is isotopic to the unknot. **[7]**



We begin by giving a twist box presentation of the knot in Figure 1.

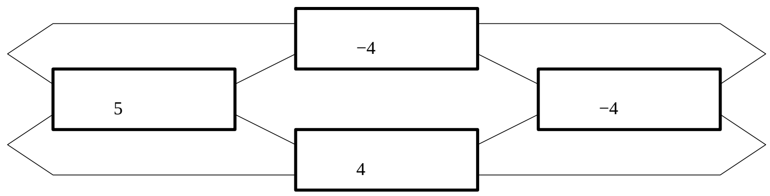


Figure 1:

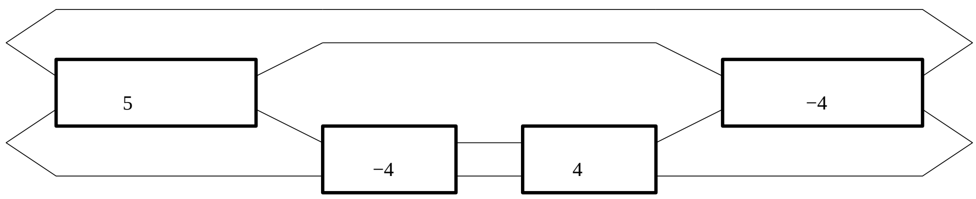
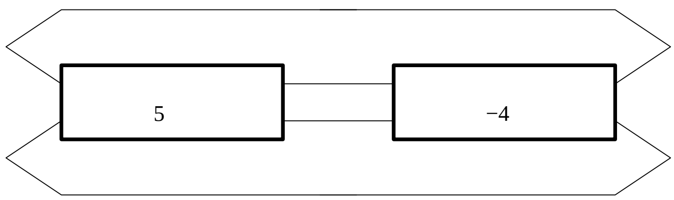


Figure 2:

Flyping the box on the left moves the top box down, as shown in Figure 2.

Adjacent twist boxes add, yielding:



Again, adjacent twist boxes add, and we are done.

e) Define the *numerator* $N(T)$ of a tangle T. **[3]**

This is best done with a picture: suppose that the four endpoints of the tangle are at the NW, NE, SE, SW points of the ball containing the tangle. Then $N(T)$ is the link formed by connecting the NE and NW points and the SE and SW points by arcs contained in the projection plane. No new crossings occur.

f) Draw the rational tangle 23/10. Draw $N(T)$. Compute $\det(N(T))$. **[6]**

We begin by computing the continued fraction expansion:

$$23/10 = 2 + \frac{1}{10/3} = 2 + \frac{1}{3 + 1/3}.$$

The rational tangle and its numerator are shown in Figure 3.

By a theorem proven in class, the determinant of $N(T)$ is the numerator of the corresponding rational number and so is 23.

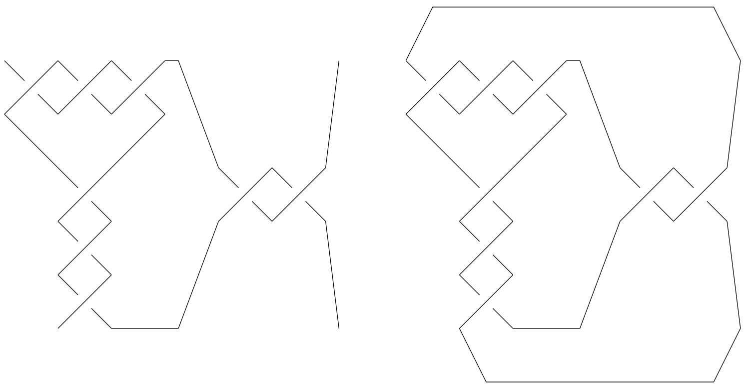


Figure 3: