MA4J2: Three Manifolds

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Lecture 1

One goal of topology is to classify manifolds up to homeomorphism. In dimension $n \geq 4$, this problem is undecidable; no algorithm, given two manifolds as an input, can decide whether or not they are homeomorphic.* We will classify manifolds in dimensions 0, 1 and 2 in the next few pages. The general topic is to classify 3–manifolds.

Definition 1.1. An *n–manifold* M^n is a Hausdorff topological space with a countable basis and such that every point $p \in M$ has an open neighbourhood U which is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n \geq 0\}$.

Remark. $\mathbb{R}^n_+$ is called the upper half space, and $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n \leq 0\}$ is called the lower half space.

Definition 1.2. ∂M is the set of points p in M such that no neighbourhood of p is homeomorphic to $\mathbb{R}^n$.

Proposition 1.1. *∂M is an $(n-1)$–manifold, and $\partial\partial M = \varnothing$.*

Definition 1.3. $\operatorname{int}(M) = M - \partial M$.

Definition 1.4. We use $I = [0,1] \subseteq \mathbb{R}$, $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and $\mathbb{D}^2 = \mathbb{B}^2$.

Definition 1.5. We give several equivalent definitions of the *sphere*:

(i) A submanifold definition: $S^n = \partial\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

(ii) A *one-point compactification* definition: S^n is the one-point compactification of $\mathbb{R}^n$, that is $S^n = \mathbb{R}^n \cup \{\infty\}$ topologized such that for any compact $K \subseteq \mathbb{R}^n$, the set $(\mathbb{R}^n - K) \cup \{\infty\}$ is a neighbourhood of ∞. Note here that $\mathbb{B}^n$ is the one-point compactification of $\mathbb{R}^n_+$.

(iii) A gluing definition: $S^n = \mathbb{B}^n_0 \sqcup \mathbb{B}^n_1 / \sim$ where $(x,0) \sim (x,1)$ if and only if $x \in \partial\mathbb{B}^n$. For example, S^1 can be obtained by joining two copies of $\mathbb{B}^1$ by their boundaries, and similarly for S^2 and $\mathbb{B}^2$.

*This result is due to A.A. Markov (1958).

Definition 1.6. We now give several equivalent definitions of *projective spaces*:

(i) A covering space definition: $\mathbb{P}^n = S^n / \sim$ where $x \sim -x$, taking S^n as in definition (i) above.

(ii) A gluing definiton: $\mathbb{P}^n = \mathbb{B}^n / \sim$ where $x \sim -x$ if and only if $x \in \partial \mathbb{B}^n$.

(iii) A moduli space definition:

$$\mathbb{P}^n = \{L \subseteq \mathbb{R}^{n+1} : L \text{ is a line through the origin}\} = (\mathbb{R}^{n+1} - \{0\}) / \sim$$

where $x \sim \lambda x$ for $\lambda \in \mathbb{R} - \{0\}$.†

Definition 1.7. We have three equivalent definitions of *tori*:

(i) A *Cartesian product* definition: $\mathbb{T}^n = (S^1)^n$, taking the Cartesian product.

(ii) A covering space definition: $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}^n / \sim$ with $x \sim y$ if and only if $x - y \in \mathbb{Z}^n$.

(iii) A gluing definition: $\mathbb{T}^n = I^n / \sim$ where $(x, 0, y) \sim (x, 1, y)$ if and only if $x \in I^k$ and $y \in I^{n-k-1}$ for any $k \in \{0, ..., n-1\}$.

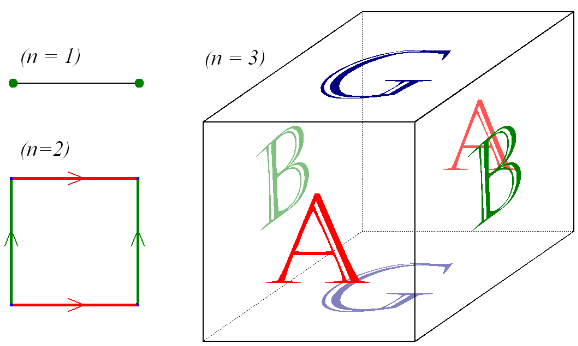

Figure 1: Construction of the first three n–tori $\mathbb{T}^n$. Identify opposite faces of I^n without twisting.

Note. $\mathbb{T}^1 \cong S^1$.

Exercise 1.1. For each set of three definitions above, prove that all three are equivalent.

In dimension zero, any compact manifold is a finite collection of points, so the classification is given by the number of points. All compact connected one-dimensional manifolds are homeomorphic to either S^1 or I.

†We will sometimes use $\mathbb{R}^*$ for $\mathbb{R} - \{0\}$.

Definition 1.8. Suppose M_i (for $i = 0, 1$) are orientable n–manifolds. Choose $\mathbb{B}_i^n \subseteq M_i$ and suppose $\varphi : \partial\mathbb{B}_0^n \to \partial\mathbb{B}_1^n$ is an orientation reversing homeomorphism. Define:

$$M_0 \# M_1 := ((M_0 - \text{int}(B_0^n)) \sqcup (M_1 - \text{int}(B_1^n)))/\sim$$

where $x \sim \varphi(x)$ whenever $x \in \partial\mathbb{B}_0^n$.

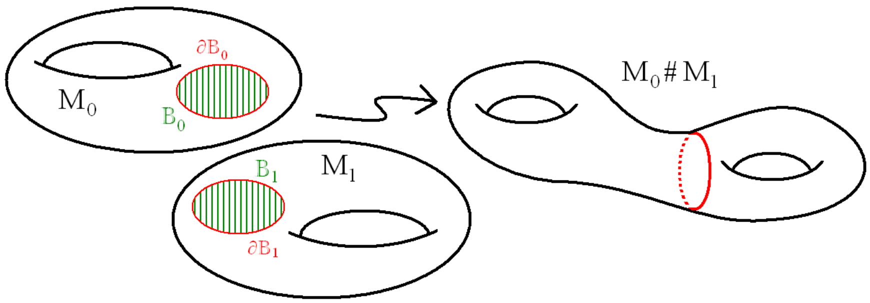

Figure 2: The connect sum. Remove the interiors of the disks $\mathbb{B}_i$ and glue along their boundaries.

Exercise 1.2. Show that $\#_3\mathbb{P}^2 \cong \mathbb{T} \# \mathbb{P}^2$.

Theorem 1.2. *Every compact connected two-dimensional manifold is homeomorphic to some $S_{g,n,c}$, where:*

$$S_{g,n,c} := (\#_g\mathbb{T}^2) \# (\#_n\mathbb{D}^2) \# (\#_c\mathbb{P}^2)$$

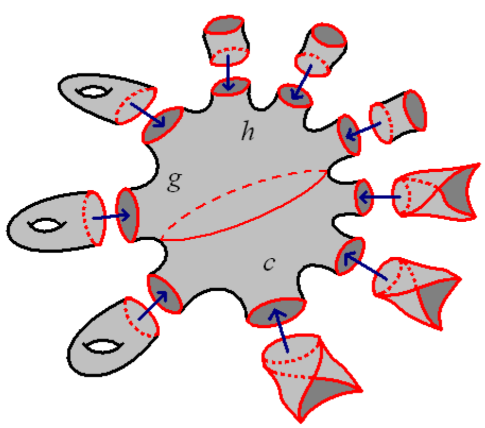

Figure 3: $S_{3,3,3}$ is the connect sum of the sphere with three tori, three Möbius strips and three 2–disks, glued along the boundary components (in red)

Example 1.1. Some spaces $S_{g,n,c}$ are homeomorphic, for example $S_{3,3,3} \cong S_{4,3,1}$.

Lecture 2

Example 2.2. We give some connect sums of three manifolds:

$$S^3 \# S^3 \cong S^3$$
$$\mathbb{T}^3 \# S^3 \cong \mathbb{T}^3$$

In general, S^n is a unit for the connect sum. $\mathbb{P}^3 \# \mathbb{P}^3$ is more interesting, as we will discuss later. On the other hand, $\mathbb{T}^3 \# \mathbb{T}^3$ invites splitting into two copies of $\mathbb{T}^3$ for a more interesting and fundamental geometry. In general, we shall find a decomposition theorem for 3–manifolds with respect to $\#$.

Definition 2.9. M^3 is *prime* if whenever $M = N \# L$ then either N or L is homeomorphic to S^3.

Remark. If $M = N \# L$ and $N \cong S^3$ then $L \cong M$, and vice versa.

Definition 2.10. M is *irreducible* if every smoothly embedded S^2 in M bounds a 3–ball.

Note. We have no examples yet of prime or irreducible 3–manifolds.

Definition 2.11. Suppose $X, Y \subseteq Z$. We say X is *ambient isotopic* (*diffeotopic*) to Y if there exists a continuous (smooth) map $F : Z \times I \to Z$ such that, defining $F_t(z) := F(t, z)$:

(i) For all $t \in I$, F_t is a homeomorphism (diffeomorphism).

(ii) $F_0 = \mathrm{Id}_Z$.

(iii) $F_1|X : X \to Y$ is a homeomorphism (diffeomorphism).

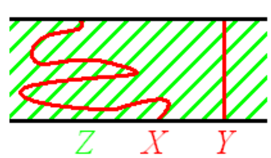

Figure 4: Here, X is ambient isotopic to Y in Z.

Theorem 2.3 (Alexander)**.** *Every smoothly embedded $S^2 \subset S^3$ is ambient isotopic to the equator.*

Compare this to:

Theorem 2.4 (Jordan-Schoenflies)**.** *Every smoothly embedded $S^1 \subset S^2$ is ambient isotopic to the equator.*

We will prove Alexander's theorem later, but for now give the following corollary.

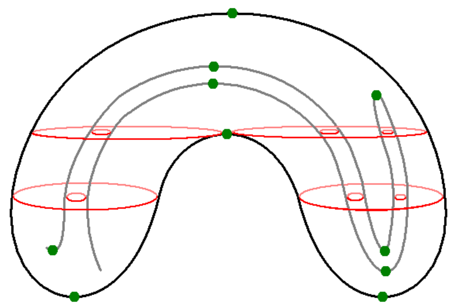

Figure 5: It is not always obvious which ball a sphere bounds

Corollary 2.5. S^3 *is prime.*

Proof. Suppose $S^3 = M \#_S N$. By Alexander's theorem, S is ambient isotopic to a round embedding of S^2 in S^3 (say the equator). Thus $M - \mathrm{int}(\mathbb{B}^3) \cong N - \mathrm{int}(\mathbb{B}^3) \cong \mathbb{B}^3$, and hence $M \cong N \cong S^3 \cong \mathbb{B}^3 \cup_\partial \mathbb{B}^3$. □

It is important that the embedding is smooth, as the following result shows.

Theorem 2.6. *There exists a topological* $S^2 \subset S^3$ *which does not bound* $\mathbb{B}^3$ *on either side.*

Note. This is a generalization of the Alexander horned sphere.

Remark. The statement of Alexander's theorem with $S^2 \subset S^3$ replaced by $S^3 \subset S^4$ is an open problem, although it has been proved that a smoothly embedded $S^3 \subset S^4$ bounds a topological ball. Brown has proved the more general statement that a smoothly embedded $S^{n-1} \subset S^n$ bounds a topological ball.

Remark. It is worth making explicit the various categories involved:

(i) Topological (TOP).

(ii) Piecewise linear (PL).

(iii) Smooth (DIFF).

These categories are all equivalent in dimension at most 3, so we move between them freely.

Exercise 2.3.

(i) Prove that any irreducible manifold is prime.

(ii) Prove that M is orientable and $S \subset M$ is a non-separating 2–sphere, then $M = N \# (S^2 \times S^1)$.

(iii) Suppose M is orientable. Then M is prime and reducible if and only if $M \cong S^2 \times S^1$. Prove the forward direction.

(iv) State and prove analogous statements to (ii) and (iii) for non-orientable manifolds.

We give one more corollary to Alexander's theorem:

Corollary 2.7. *If $M \subseteq S^3$ is compact and has $|\partial M| \leq 1$ (at most one boundary component) then M is irreducible.*

Example 2.3. We give further examples of irreducible manifolds. Suppose $K \subset S^3$ is a knot, that is a smooth embedding of S^1. Let $N(K) \subseteq S^3$ be a closed regular neighbourhood (i.e. a tubular neighbourhood) of the knot. Let $n(K) = \mathrm{int}(N(K))$. Then the knot exterior $X_K := S^3 - n(K)$ is irreducible, by the previous corollary.

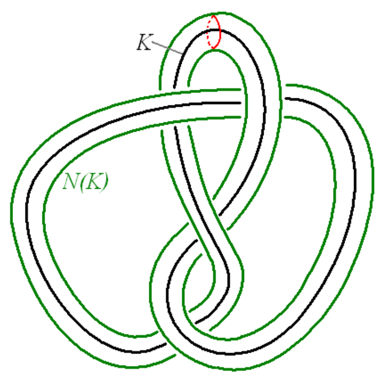

Figure 6: A tubular neighbourhood of the figure 8 knot

Lecture 3

We now prove Alexander's theorem. More precisely, we will prove that any (smoothly) embedded $S^2 \subset \mathbb{R}^3$ bounds a 3-ball, from which the theorem can be deduced as a corollary.

Exercise 3.4. Show how Alexander's theorem follows from this statement.

We need the following lemma:

Lemma 3.8. *Suppose that a manifold M^n and $\mathbb{B}_1^{n-1} \subseteq \partial M^n$ are given, as is a diffeomorphism $\varphi : \mathbb{B}_0^{n-1} \to \mathbb{B}_1^{n-1}$, where $\mathbb{B}_0^{n-1} \subseteq \partial \mathbb{B}^n$. Then $M^n \cup_\varphi \mathbb{B}^n \cong M^n$, as per Figure 7.*

As a consequence, if B and B' are n–balls, then $B \cup_\partial B'$ is a ball (Figure 8(a)), as is $\overline{B - B'}$ if $B' \subset B$ and $\partial B' \cap \partial B \cong \mathbb{D}^{n-1}$ (Figure 8(b)).

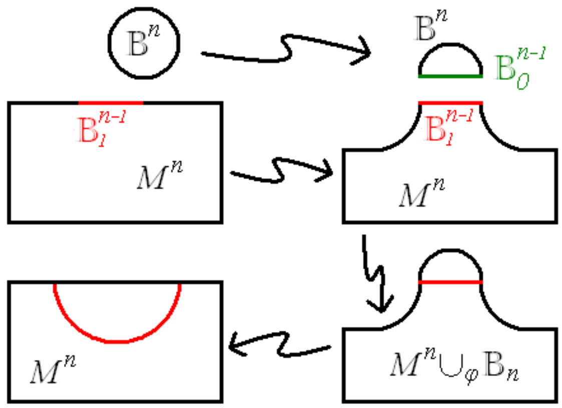

Figure 7: Glueing M^n to $\mathbb{B}^n$ along submanifolds of their boundaries is homeomorphic to M^n.

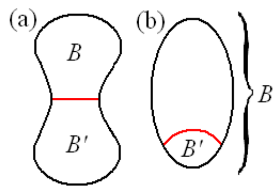

Figure 8: (a) B, B' balls $\Rightarrow B \cup_\partial B'$ a ball, and (b) $B' \subset B$ and $\partial B' \cap \partial B \cong \mathbb{D}^{n-1} \Rightarrow \overline{B - B'}$ a ball.

Theorem 3.9. *Any smoothly embedded $S^2 \subset \mathbb{R}^3$ bounds a 3–ball.*

Proof. Suppose $S^2 \cong S \subset \mathbb{R}^3$ is smooth. We can isotope S so that $z : S \to \mathbb{R}$ (the height function, giving the z co-ordinate) is a Morse function. Thus all critical points are of the standard three types; cups (minima), caps (maxima), and saddles, and all critical points occur at distinct heights (as illustrated in Figure 9). Choose $a_i \in \mathbb{R}$ such that $(-\infty, a_1), (a_1, a_2), ..., (a_{n-1}, \infty)$ each contain

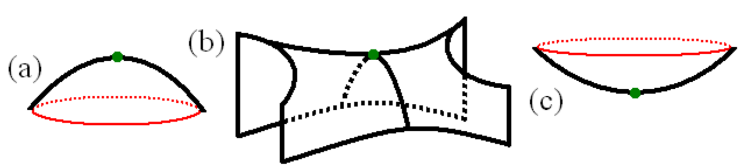

Figure 9: (a) A cap. (b) A saddle. (c) A cup.

exactly one critical value, as in Figure 10. Let:

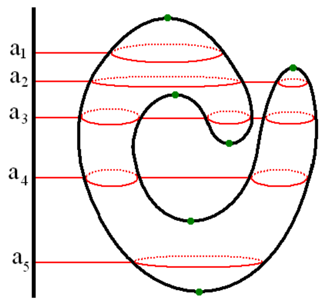

Figure 10: The red circles are regular values separating the critical points (green). Here we have $(n, w) = (6, 9)$.

$$L[a, b] := \{(x, y, z) : z \in [a, b]\}$$
$$L(a) := \{(x, y, z) : z = a\}$$
$$L_i := L(a_i)$$

Define $n(S)$ to be the number of critical points. Define the *width* by:

$$w(S) = \sum_{i=1}^{n-1} |S \cap L_i|$$

This is the number of red circles in Figure 10. We will induct on $(n(S), w(S))$ lexicographically. Note that the components of $L_i \cap S$ are all simple closed curves, because each a_i is a regular value. So by the Jordan-Schoenflies theorem, they all bound disks. Say that β, a component of $L_i \cap S$, is *innermost* if D_β, the disk bounded by β, has the property that $D_\beta \cap S = \beta$. Notice that β also bounds a pair of disks in S. Label a_i with an A (resp. B) if there is some

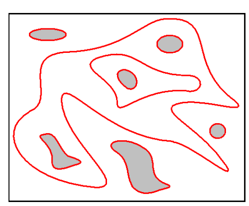

Figure 11: The intersection of the plane L_i with the sphere. Shaded components are innermost.

innermost curve $\beta \subseteq L_i \cap S$ such that one disk of $S - \beta$ contains exactly one critical point, a maximum (resp. minimum). Note that a_i could receive both labels. Note also that a_1 is labelled by B and a_{n-1} is labelled by A. We have cases:

Case 1: Some a_i is labelled both A and B.

Case 2: Some a_i is unlabelled.

Case 3: There exists i such that a_i is labelled B and a_{i+1} is labelled A.

Exercise 3.5. Check that we must always be in at least one of these cases.

We prove these in turn:

Case 1a: Some innermost $\beta \in L_i \cap S$ bounds a disk in S above and bounds a disk in S below, each with one critical point; this forms the base case of the induction, where $n(S) = 2$ and $w(S) = 1$. We claim that in this case S bounds a ball. To see this, cut off the two critical points with planes

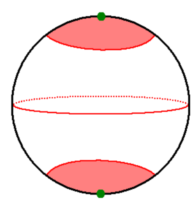

Figure 12: The base case.

slightly above the minimum and below the maximum, removing two 3–balls from S, and giving a compact cylinder. We claim that for every $a \in \mathbb{R}$ such that the set $L(a)$ intersects this compact cylinder, there exists $\varepsilon > 0$ such that $S \cap L[a, a+\varepsilon]$ bounds a 3–ball in $L[a, a+\varepsilon]$. This can be proved by the implicit function theorem and the isotopy extension theorem. See Hatcher's *Notes on basic 3–manifold topology* for more details. Note that

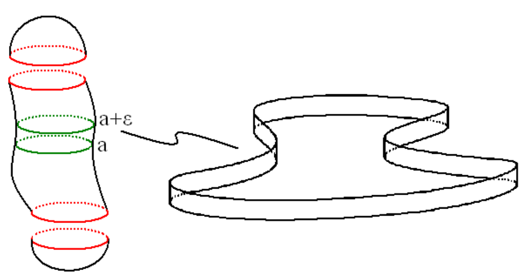

Figure 13: The slab bounded by $L_[a, a+\varepsilon]$.

the intersection $L(a) \cap S$ is a curve, so bounds a disk. Note that finitely many of the $L[a, a+\varepsilon]$ cover the compact cylinder. Glue these slabs together, and re-attach the cap and cup. By Lemma 3.8, this gives a 3–ball.

This proof continues in the lectures from week two.