MA4J2 Three Manifolds

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Lecture 28

Definition 28.1. Suppose M, N are 3–manifolds and $D \subset \partial M$ and $E \subset \partial N$ are disks. Let $\varphi\colon D \longrightarrow E$ be an orientation reversing homeomorphism. Then we define the *boundary connect sum* of M and N to be $M \#_\partial N := M \sqcup N/\varphi$. See Figure 1.

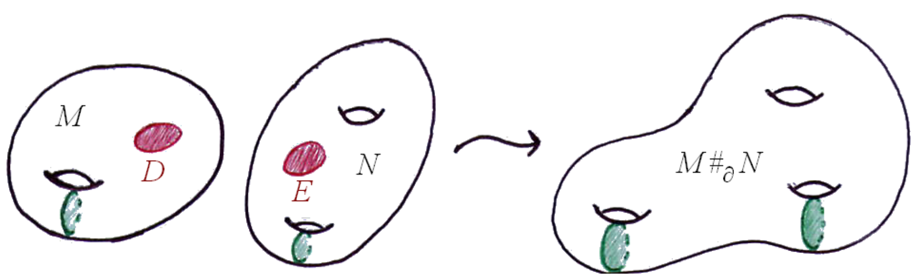

Figure 1: An example of the boundary connect sum.

Recall that φ only matters up to isotopy.

Definition 28.2. Suppose V is a handlebody and $F = \sqcup F_i$ is a collection of closed orientable surfaces, none of which is a two-sphere. Then $C := V \#_\partial (\#_\partial F_i \times I)$ is a *compression body*. We define the *inner boundary* $\partial_- C = \sqcup_{F_i \times \{0\}} C$ and the *outer boundary* $\partial_+ C = \partial C - \partial_- C$.

Example 28.1. See Figure 2.

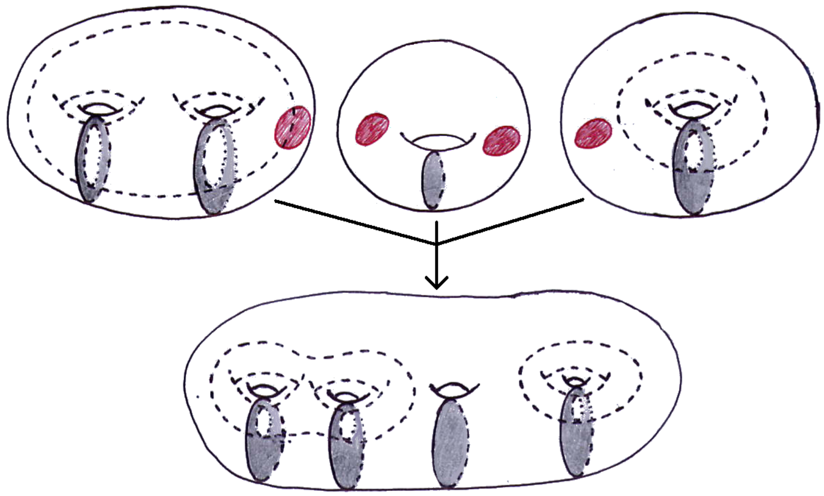

Figure 2: Another example of the boundary connect sum. Note that the third grey surface is a disk while the others are all annuli.

Exercise 28.1. Show that $\#_\partial$ is associative, commutative and B^3 is the unit.

Exercise 28.2. Show that the essential surfaces in C are

- essential disks compressing $\partial_+ C$,
- components of $\partial_- C$ and
- annuli meeting both $\partial_+ C$ and $\partial_- C$.

Example 28.2. See Figure 3.

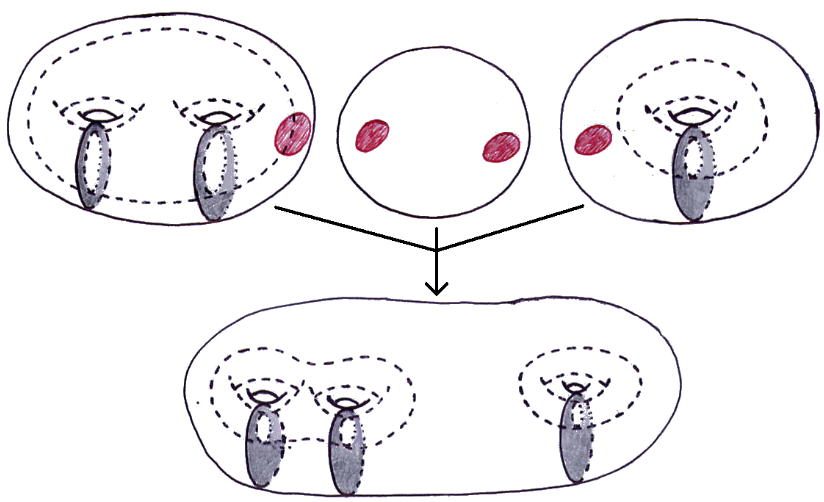

Figure 3: An example of the boundary connect sum.

Now we demonstrate the existence of short hierarchies, following Jaco. Suppose that M_0 is Haken and additionally that ∂M_0 is incompressible. Let $S_0 \subset M_0$ be a maximal collection of disjoint, non-parallel, closed, incompressible, two-sided surfaces in M_0 none of which are spheres. Since M_0 is Haken, S_0 is non-empty and it is finite by Haken-Kneser finiteness. See Figure 4.

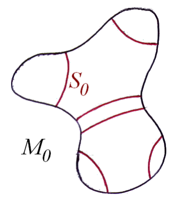

Figure 4: $S_0 \subset M_0$ is non-empty and finite. It is convenient to take $\partial M_0 = \varnothing$.

Aside. Note that closed incompressible surfaces, which are not spheres, are essential.

Note that every component $N \subset M_1 := M - n(S_0)$ has boundary with genus ≥ 1. So N contains some essential surface by Theorem 27.5. Let $S_1 \subset M_1$ be a maximal collection of disjoint, nonparallel, two-sided, essential surfaces in M_1: these are the green lines in Figure 5. Again, S_1 cuts every component of M_1 and S_1 is finite by Haken-Kneser finiteness in the bounded case. See the addedum to Exercise 5.5. Define $M_2 := M_1 - n(S_1)$ and let C be any component of M_2.

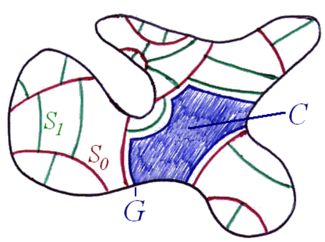

Figure 5: The component C contains an essential surface.

Proposition 28.1. *The component C is a compression body.*

Proof. Suppose that some component $G \subset \partial C$ is compressible into C. So let G_i, D_i be a sequence where $G_0 = G$ and D_i compresses G_i in the same direction as D_0, into C. Define $G_{i+1} = (G_i)_{D_i}$. So we get a sequence

$$G_0 \xrightarrow{D_0} G_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{n-1}} G_n.$$

See Figure 6.

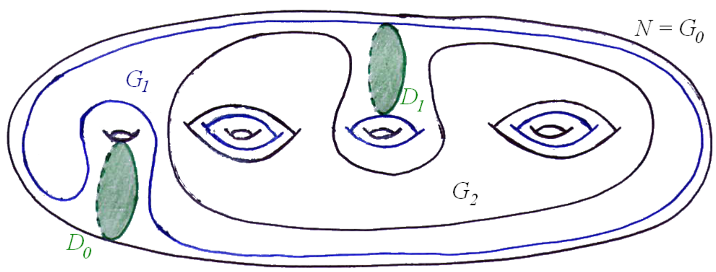

Figure 6: The first few terms in the sequence (G_i, D_i).

Note that G_{i+1} may be disconnected, as in Figure 7.

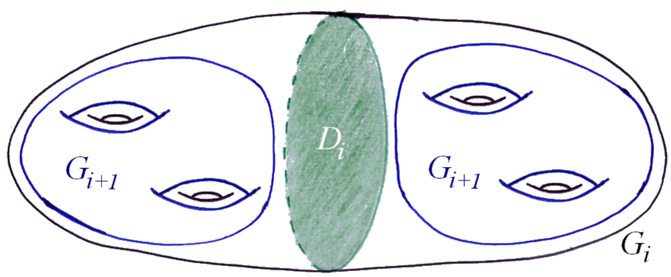

Figure 7: G_{i+1} may be disconnected.

Claim. If some component of G_n is a 2–sphere then it bounds a 3–ball in C.

Proof sketch. M is irreducible, thus C is irreducible as well. □

So cap off such 2–spheres, deleting them from G_n.

Claim. The closed surface G_n is incompressible in M.

Proof. As G_n is last in the sequence, G_n cannot compress into C. So suppose E is a surgery disk for G_n in the other direction. See Figure 8.

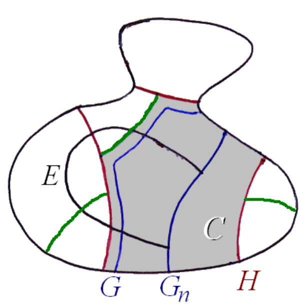

Figure 8: E is a compressing disk for G_n in the other direction. C is all of the grey area.

Then we can do the following: Isotope E off of S_0, then off of S_1 and then off of $\{D_i\}$. It follows that E is a surgery disk for G_n in the compression body cobounded by G_0 and G_n. Thus G_n is the inner boundary of this compression body and so is essential. Thus E is trivial, as desired. □

To finish the proposition, deduce that the components of G_n are parallel to components $H \subset S_0$ since G_n is essential, closed and disjoint from S_0 (as it lies in C). Again see Figure 8. □

Now let $S_2 \subset M_2$ be a collection of essential disks, cutting all compression bodies into products. Let $S_3 \subset M_3$ be a collection of vertical annuli (one per product). Finally $S_4 \subset M_4$ is a collection of disks cutting all handlebodies into 3–balls, as in Figure 9. This proves the existence of *short hierarchies.*

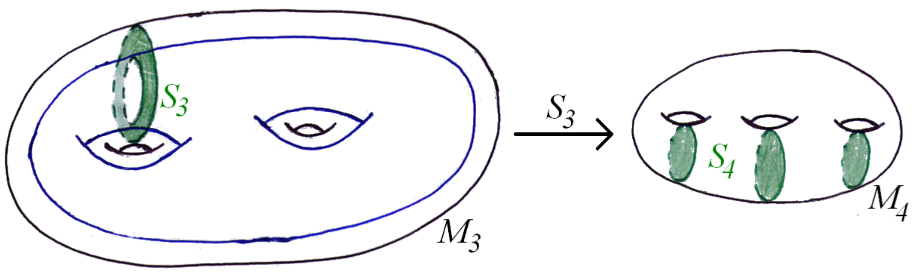

Figure 9: S_3 is a collection of vertical annuli; cut along these annuli to get a collection of handlebodies. Then cutting along S_4 gives a collection of 3-balls.

Lecture 29

In this lecture, we again follow Lackenby.

Definition 29.3. A *boundary pattern* P for M^3 is a trivalent graph embedded in ∂M. We allow P to be the empty set, to be disconnected and to have simple closed curves as components.

Example 29.3. Trivalent graphs in $S^2 = \partial B^3$ are patterns for B^3. See Figure 10.

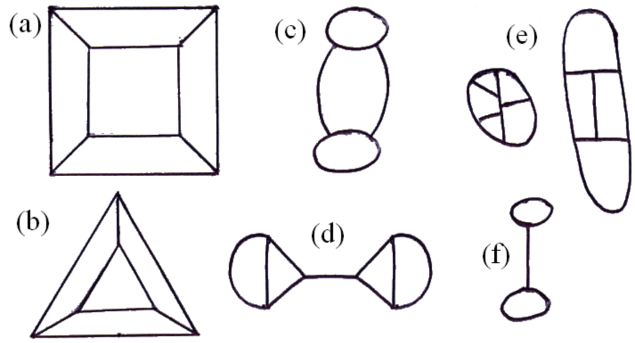

Figure 10: Six examples of trivalent graphs in S^2. Note that (e) is a disconnected pattern.

Suppose (M, P) is a manifold equipped with a boundary pattern. Suppose $S \subset M$ is properly embedded and ∂S is transverse to P. So ∂S misses the vertices of P and intersects the edges of P transversely. Let $N = M - n(S)$ and let

$$Q = (P - n(S)) \cup \partial S^+ \cup \partial S^-.$$

So Q is a pattern for N and we write $(M, P) \xrightarrow{S} (N, Q)$. See Figure 11.

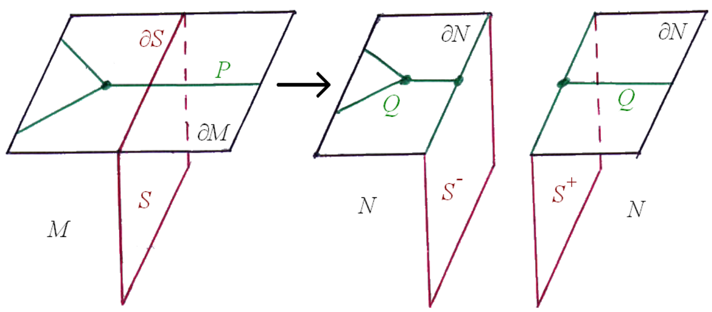

Figure 11: A picture of the cutting.

Definition 29.4. Let P be a boundary pattern for M. Then we call P *essential* if for any $(D, \partial D) \subset (M, \partial M)$ with ∂D transverse to P and $|\partial D \cap P| \leq 3$ we have

- a disk $E \subset \partial M$ such that $\partial E = \partial D$ and
- the intersection $E \cap P$ contains at most one vertex of P and contains no cycles of P.

Exercise 29.3. Verify that if P is essential then we get the implications shown in Figures 12 to 15:

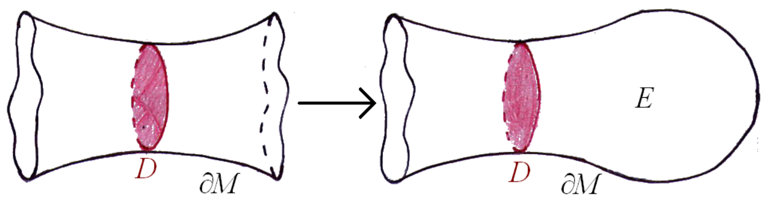

Figure 12: The case $\partial D \cap P = \varnothing$.

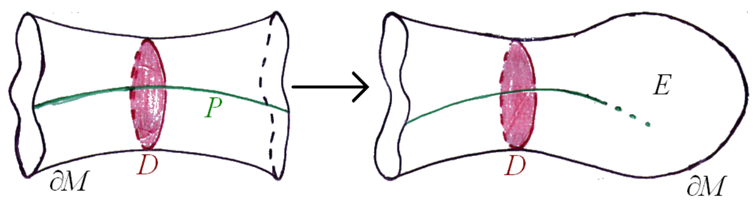

Figure 13: The case $|\partial D \cap P| = 1$ is not possible.

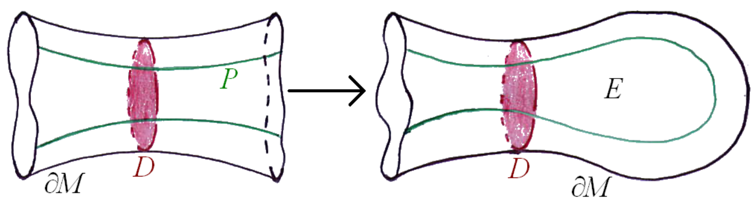

Figure 14: The case $|\partial D \cap P| = 2$.

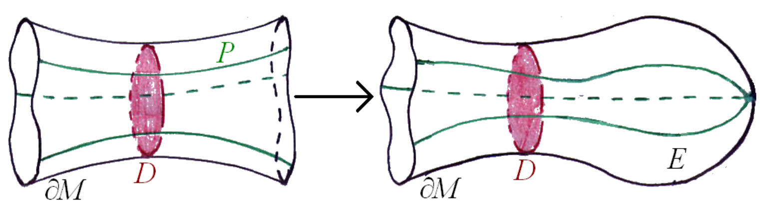

Figure 15: The case $|\partial D \cap P| = 3$.

Exercise 29.4. Analyse the examples of (B^3, P) given above. Which are, and which are not, essential?

Exercise 29.5. Give necessary and sufficient conditions for P to be an essential pattern for B^3.

Example 29.4. If $M_0 = \mathbb{T}^3 = I^3/\sim$ then $S_0 = \{z = 0\}$ is an essential torus, $S_1 = \{x = 0\} \subset M_1$ is an essential annulus and $S_2 = \{y = 0\} \subset M_2$ is an essential disk. We can see this in Figure 16.

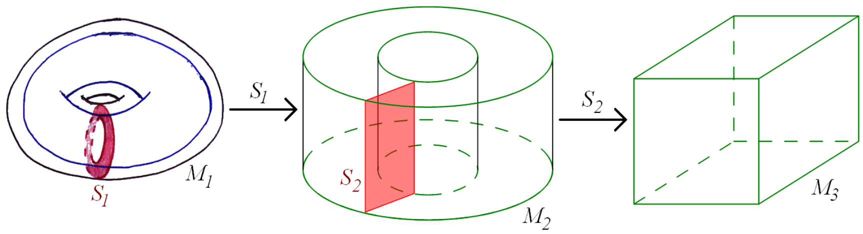

Figure 16: Pictures of these cuttings with boundary patterns. For M_3, P_3 is the 1–skeleton of the cube.

Definition 29.5. Let $P \subset \partial M$ be a pattern. We say P is *homotopically essential* if the following condition hold. For any map $f\colon (D, \partial D) \longrightarrow (M, \partial M)$ (which need not be an embedding) transverse to P, we define $Z = Z_f = \partial D \cap f^{-1}(P)$. If $|Z| \leq 3$ then there is a homotopy $H\colon D \times I \longrightarrow M$ such that

- for all t: $H_t|Z = f|Z$,
- $H_0 = f$,
- $H_1(D) \subset \partial M$ and finally
- $H_1(D)$ contains at most one vertex of P and contains no cycles of P.

Exercise 29.6. If P is homotopically essential, then P is essential.

Theorem 29.1 (9.1 in Lackenby)**.** *If P is essential, then it is homotopically essential.*

We will indicate a proof, using *special hierarchies*, in the next lecture.

Exercise 29.7. Theorem 29.1 implies the Disk Theorem. As a hint, recall that we allow $P = \varnothing$.

Lecture 30

We pause to give another example of a hierarchy.

Example 30.5. Consider the knot $K \subset S^3$ shown in Figure 17: the $(1, 1, -3)$–pretzel knot. The surface shown is a spanning surface for K. This is one of the two so-called *checkerboard surfaces* for this diagram of K.

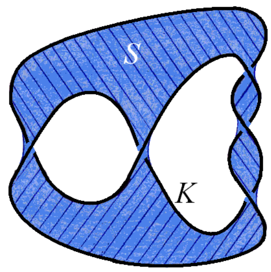

Figure 17: A diagram of the $(1, 1, -3)$–pretzel and S, one of its two checkerboard surfaces.

Near a twist we see a half-twisted band, as in Figure 18.

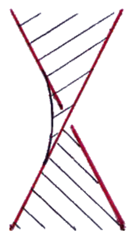

Figure 18: A half twisted band.

Let $N = N(K)$ be a regular neighbourhood and write $X = X_K = S^3 - n(K)$. See Figure 19. Let S_0 be the remains of the spanning surface in X.

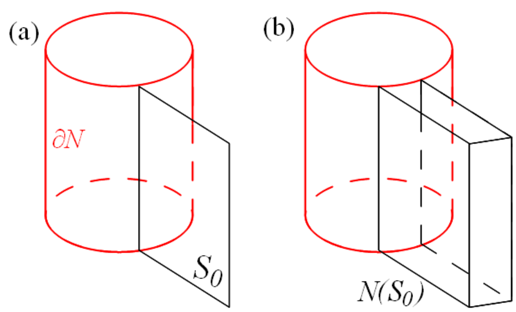

Figure 19: (a) A picture of $N(K)$, S_0 and (b) $N(S_0)$.

Let $M_0 = X$ and cut M_0 along S_0 to get M_1. Thus, as M_1 is a genus two handlebody, we find that $\partial S_0^{\pm}$ gives a pattern to ∂M_1, shown in Figure 20.

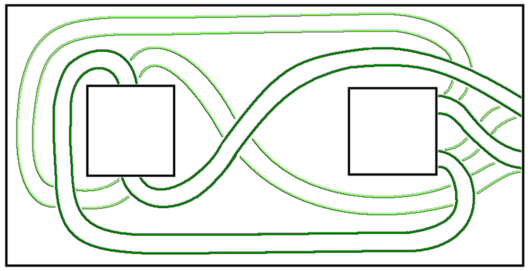

Figure 20: A pattern to ∂M given by $\partial S^{\pm}$. Note that M_1 is the handlebody on the outside.

The two components of P in ∂M_1 cobound an annulus, the remains of ∂N. We take S_1 to be the union of a pair of disks as in Figure 21.

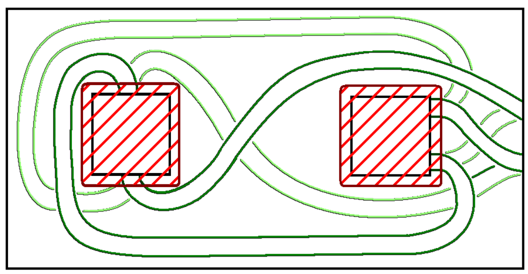

Figure 21: The essential surface S_1 in M_1, consisting of two disks which meet ∂M_1 in two loops around the holes.

Now cut along S_1 to get $M_2 \cong B^3$.

Exercise 30.8. Show that (M_2, P_2) is homeomorphic to the pattern shown in Figure 22.

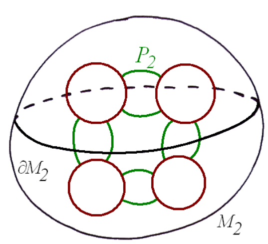

Figure 22: A 3–ball with a pattern.

Exercise 30.9. Show that $P_2 \subset \partial M_2$ is essential. Figure 23 may be helpful.

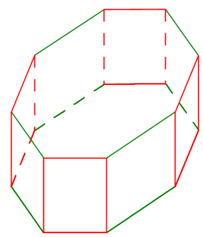

Figure 23: $(M_2, P_2) \cong \text{Oct} \times I$ where Oct denotes an octagon.

Claim. The surface $S_0 \subset X$ is essential.

Proof. Suppose $(D, \partial D) \subset (X, S_0)$ is a surgery disk. So consider $D \cap S_1 \subset D$. This is a collection of simple loops and arcs.

1. Suppose α is an innermost loop. Then α bounds E in D. So $(E, \alpha) \subset (M_2, \partial M_2)$ and $\alpha \cap P_2 = \varnothing$ which implies that we may isotope E past S_1, reducing $|S_1 \cap D|$. See Figure 24.

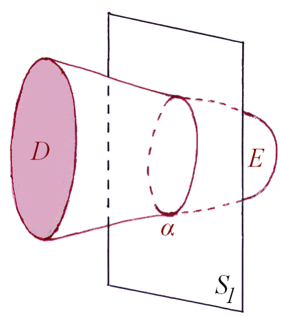

Figure 24: We may isotope E past S_1, reducing $|S_1 \cap D|$.

2. Suppose $\alpha \subset D$ is an outermost arc of $S_1 \cap D$. So α cuts off a bigon E. So $(E, \partial E) \subset (M_2, \partial M_2)$ is a bigon and $\partial E \cap P_2$ is exactly two points. But (M_2, P_2) is essential and we continue as usual.

So we may assume that $D \cap S_1 = \varnothing$. So $(D, \partial D)$ embeds in $(M_2, \partial M_2)$ with $\partial D \cap P_2 = \varnothing$. Since M_2 is a ball we find that D is parallel to a disk $D' \subset S_0$. So S_0 is incompressible. Now by Lemma 20.2 (1.10 in Hatcher) S_0 is boundary incompressible. It is also possible to directly prove that by repeating the proof using bigons. See Figure 25. □

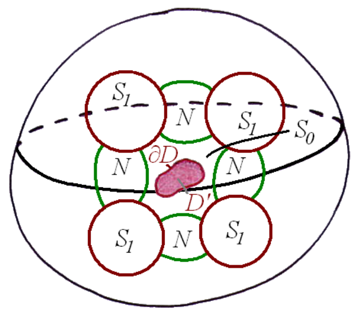

Figure 25: D is parallel to a disk $D' \subset S_0$.

We now give the ideas necessary to prove Theorem 29.1. We need a few more definitions.

Definition 30.6. Suppose $S \subset (M, P)$ is properly embedded and suppose $P \subset \partial M$ is an essential pattern. A surgery bigon D for S is a *pattern surgery* if $|\beta \cap P| \leq 1$ where $\partial D = \alpha \cup \beta$ and $\alpha = \partial D \cap S$. Say D is *trivial* if α cuts a bigon E out of S with $\partial E = \alpha \cup \gamma$ and $|\gamma \cap P| \leq 1$. Otherwise call D a *pattern compression.*

Definition 30.7. If S is essential and all pattern surgeries are trivial, we call S *pattern essential.* See Figure 26.

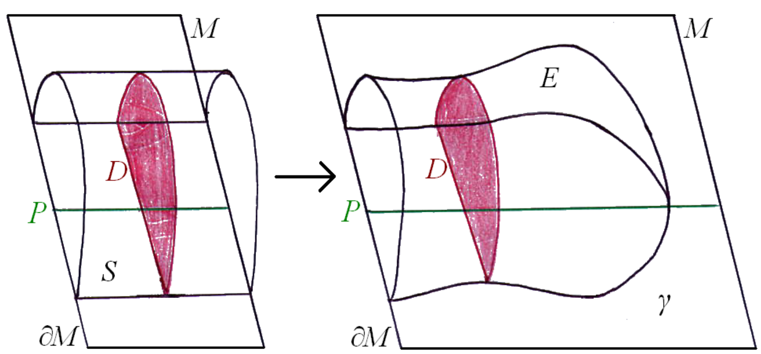

Figure 26: A picture of what it means to be pattern essential.

Definition 30.8. A *special hierarchy* is a sequence $(M_i, P_i) \xrightarrow{S_i} (M_{i+1}, P_{i+1})$ where all P_i are essential and all S_i are pattern essential. We do not allow S_i to be a sphere.

Proposition 30.2. *If $S \subset (M, P)$ is essential we may isotope S to be pattern essential.*

Proof. Exercise. □

Using the above one can show the following two propositions which imply Theorem 29.1.

Proposition 30.3. *If P is a pattern for $M \cong B^3$ and is essential, then P is homotopically essential.* $\square$

Proposition 30.4. *If $(M, P) \xrightarrow{S} (N, Q)$ are all essential and $Q \subset \partial N$ is homotopically essential, then P is homotopically essential in M.* $\square$