MA4J2 Three Manifolds

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Lecture 4

(Proof continued from last lecture).

Case 1b There are innermost $\alpha, \beta \subset L_i \cap S$ so that α bounds D above, β bounds E below. Let D', E' be the disks bounded by α, β, inside of L_i. So, by the base case $D \cup D'$ $(E \cup E')$ bounds a 3–ball. Use this 3–ball to define an ambient isotopy that flattens D (E), pushing the critical point just below (above) the plane L_i.

Exercise 4.1. Show that this reduces $w(S)$.

Case 2 The regular value a_i is not labelled. For this case, we first have to introduce

Definition 4.1. Suppose $F^2 \subset M^2$ is properly embedded (i.e. a submanifold, i.e. embedded and $F \cap \partial M = \partial F$). We say $(D^2, \partial D) \subset (M, F)$ is a *surgery disk* for F if $D \cap F = \partial D$.

Let $n(\partial D)$ be an open annular neighbourhood of ∂D, in F. Let D_+, D_- be parallel copies of D in M. Define F *surgered along* D by $F_D := (F - n(\partial D)) \cup D_+ \cup D_-$, as in Figure 1.

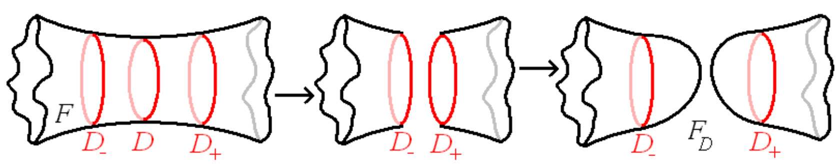

Figure 1: Surgery. $F_D := (F - n(\partial D)) \cup D_+ \cup D_-$.

We now return to case 2. Suppose $\beta \subset S \cap L_i$ is innermost. So, β bounds D above, E below, $D \cup_\beta E = S$ and D, E each contain at least 3 critical points. Say β bounds a disk $B \subset L_i$. So:

$$S_B = S_+ \cup S_-, \quad S_+ \cong D \cup B_+, \quad S_- \cong E \cup B_-.$$

Thus $n(S_+), n(S_-) < n(S)$ since $n(S_+)+n(S_-) = n(S)+2$. By induction, S_+, S_- each bound a 3–ball X_+, X_- thus so did S, applying Lemma 1.3 in Hatcher's notes. In the first case $X_+ \cap X_- = B$ and so we take the union. In the second case $X_+ \subset X_-$, we take the difference. See Figure 2.

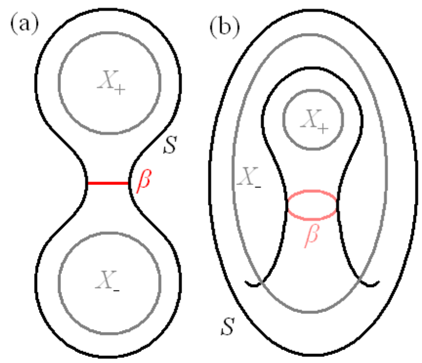

Figure 2: The case when (a) $x_+ \cap X_- = \varnothing$ or (b) $X_+ \subseteq X_-$.

Case 3 The regular value a_i is labelled only B and the regular value a_{i+1} is labelled only A. Between L_i and L_{i+1} we have $S \cap L[a_i, a_{i+1}]$ is a union of cylinders, caps, cups, pairs of pants, upside down pairs of pants and pants with inverted legs, as illustrated in Figure 3.

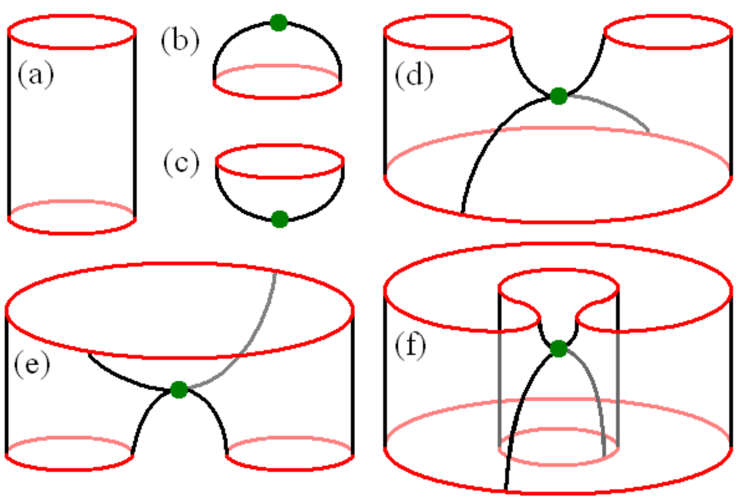

Figure 3: $S \cap L[a_i, a_{i+1}]$ is a union of (a) cylinders, (b) caps, (c) cups, (d) pairs of pants, (e) upside down pairs of pants, (f) pants with inverted legs and (g) an upside down version of (f) (not shown).

Note that there is at most one critical point in $S \cap L[a_i, a_{i+1}]$, so it is a saddle (check this using the labelling). Using the labelling deduce that either α or β is a cuff of the pants.

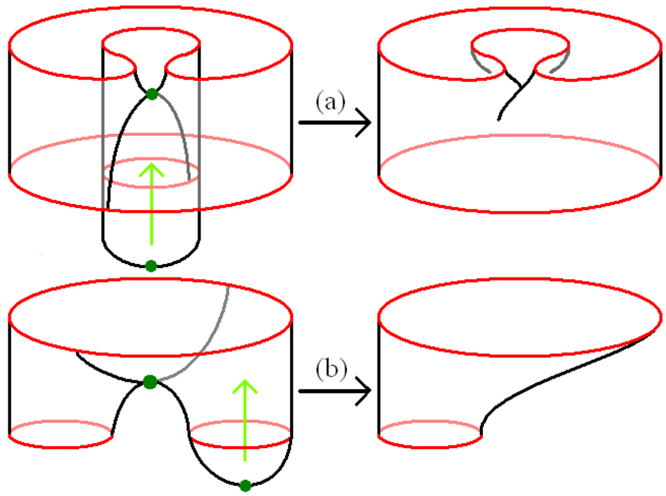

Figure 4: Two examples of how may isotope E to be in L_i and then upwards, canceling two critical points.

We have that β is innermost in L_i and β bounds (in S) a disk below, E, with a single critical point (minimum). Hence, by the base case, we may isotope E to be in L_i and then upwards to cancel two critical points, as in Figure 4. Thus, we have isotoped S to a sphere S' such that $n(S') = n(S) - 2$. This completes the induction step and so, the proof. □

Lecture 5

Definition 5.2. Say a 2–sphere $S \subset M^3$ is *essential* if no component of $M - n(S)$ is a 3–ball.

Incompressible surfaces

Definition 5.3. Suppose $F^2 \subset M^3$ is properly embedded. Suppose $(D, \partial D^2) \subset (M, F)$ is a surgery disk. Say that D is a *trivial surgery* disk if $\partial D \subset F$ is equal to $\mathbf{1} \in \pi_1(F)$ where $\pi_1(F)$ is the fundamental group of F. We say that D is a *compressing disk* if $\partial D \subset F$ is not equal to $\mathbf{1} \in \pi_1(F)$.

An alternative definition is: D is a trivial surgery disk if ∂D bounds a disk in F. See Figure 5.

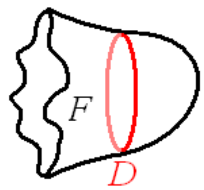

Figure 5: Here, D is a trivial surgery disc for F.

Exercise 5.2. Check that a simple closed curve $\alpha \subset F$ bounds a disk $E \subset F$ if and only if $[\alpha] = \mathbf{1} \in \pi_1(F)$.

Definition 5.4. Suppose $F \subset M$ is either proper embedded or $F \subset \partial M$ is a subsurface. Then we say that F is *compressible* if and only if there exists a compressing disk for F. Otherwise we call F *incompressible*.

Example 5.1. Let $T \subset S^3$ be the standard embedding, i.e. $\partial N(U)$ where U is the unknot. Then T is compressible since there are two compressing disks. We call them the *meridian disk* and the *longitude disk* respectively, as illustrated in Figure 6.

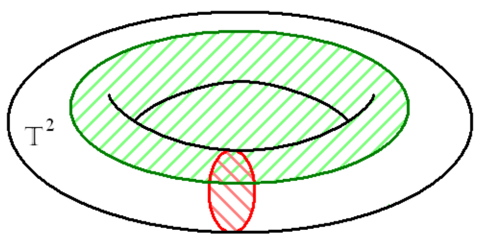

Figure 6: The meridian disk is in green while the longitude disk is red. The boundary of the meridian disk is a circle in T but its interior is in S^3.

Example 5.2. If $M = D \times S^1$ is a solid torus then $\partial M \subset M$ is compressible.

Exercise 5.3. Show that $T = \mathbb{T}^2 \times \{\frac{1}{2}\} \subset \mathbb{T}^2 \times I = M$ is incompressible.

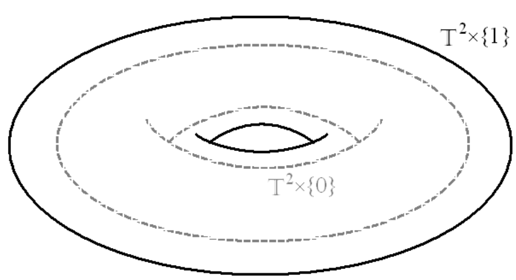

Figure 7: $T = \mathbb{T}^2 \times \{\frac{1}{2}\} \subset \mathbb{T}^2 \times I = M$ is incompressible.

Exercise 5.4. Suppose that M is an irreducible three-manifold and $F, G \subset \partial M$ are disjoint, incompressible subsurfaces. Suppose that $\varphi\colon F \longrightarrow G$ is a homeomorphism. Show that M/φ is irreducible.

Note. One can check $M = \mathbb{D}^2 \times S^1$ is irreducible but $D(M)$, the double of M, is not. Here $D(M) = M_0 \sqcup M_1/\sim$, where $(x,0) \sim (x,1)$ if and only if $x \in \partial M$ where $M_i = M \times \{i\}$.

Exercise 5.5.

1. If $F \subset S^3$ is closed, $F \neq S^2$, then F is compressible.
2. (Alexander) Any $\mathbb{T}^2 \subset S^3$ bounds a solid torus ($\mathbb{D}^2 \times S^1$) on at least one side.

Definition 5.5. Let V_g be the *handlebody* of genus g, i.e.

$$V_g = \underbrace{\mathbb{D}^2 \times S^1 \ \cup_{\mathbb{D}^2} \ \mathbb{D}^2 \times S^1 \ \cup_{\mathbb{D}^2} \ \ldots \ \cup_{\mathbb{D}^2} \ \mathbb{D}^2 \times S^1}_{g \text{ times}}$$

By convention, $V_0 = \mathbb{B}^3$. See Figure8.

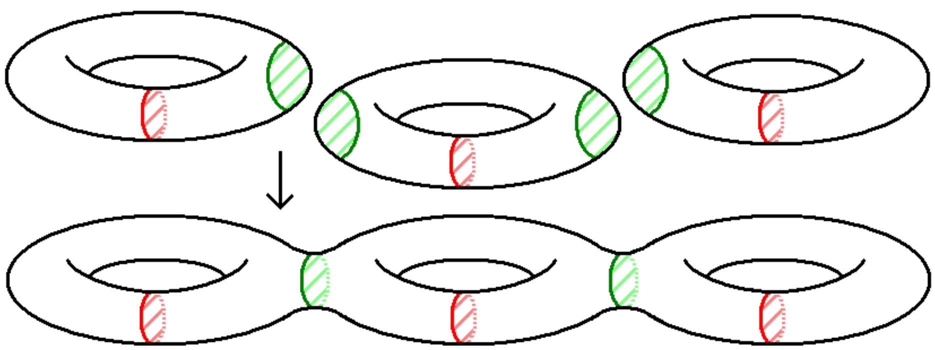

Figure 8: The handlebody V_3. Note that V_g is "solid", and not a surface.

Example 5.3. Find $S_2 \hookrightarrow S^3$ which does not bound a handlebody on either side. Here S_2 denotes a surface of genus 2.

Remark. $\partial V_g = \#_g \mathbb{T}^2 = S_g$ because $\partial(\mathbb{D}^2 \times S^1) = S^1 \times S^1 = \mathbb{T}^2$.

Products and Bundles

A map $\rho\colon Z \longrightarrow X$ is a Y*–bundle* (or a *fibre bundle*) if for all $x \in X$ there exists a neighbourhood $x \in U \subset X$ and a homeomorphism $h_U\colon Y \times U \longrightarrow \rho^{-1}(U)$ such that the composition $\rho \circ h_U$ is the projection onto the second coordinate. Here, Z is called the *total space*, X the *base space*, Y the *fibre* and h_U is called a *local trivialization*.

Example 5.4. Let $Z = \mathbb{D}^2 \times S^1$ and denote ρ_i be the projection onto the i–th coordinate. Then $\rho_1 \colon Z \longrightarrow \mathbb{D}^2$ is a S^1–bundle map and $\rho_2 \colon Z \longrightarrow S^1$ is a $\mathbb{D}^2$–bundle map.

Lecture 6

Bundles and Neighbourhoods

See Lackenby §6.

Definition 6.6. We say $Z \xrightarrow{\rho} X,\ Z' \xrightarrow{\rho'} X$ are *equivalent* Y–bundles if there is a homeomorphism $h \colon Z' \longrightarrow Z$ making the following diagram commute

$$\begin{array}{ccc} Z' & \xrightarrow{h} & Z \\ {\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{\mathrm{Id}_X} & X \end{array}$$

Corollary 6.1 (See Corollary 6.3 in Lackenby's notes)**.** *If X is contractible then any Y–bundle $Z \xrightarrow{\rho} X$ is equivalent to the product bundle $Y \times X \xrightarrow{\rho_2} X$.*

Exercise 6.6. Prove this directly for $X = \mathbb{B}^1, \mathbb{B}^2$.

Exercise 6.7. Find a S^1–bundle over S^2 that is not equivalent to the product bundle. It follows that the fundamental group $\pi_1(X, x) = \{\mathbf{1}\}$ is not sufficient hypothesis for Corollary 6.1.

Lemma 6.2 (See Lemma 6.4 in Lackenby's notes)**.** *For all $n \in \mathbb{N}$ there are exactly two $\mathbb{B}^n$–bundles over S^1 up to equivalence. These are*

- *the trivial bundle $\mathbb{B}^n \times S^1$*
- *the twisted bundle $\mathbb{B}^n \widetilde{\times} S^1 = \mathbb{B}^n \times I / (x, 0) \sim (r(x), 1)$, where $r(x_1, \ldots, x_n) = (x_1, \ldots, -x_n)$ is a reflection.*

Version of the Tubular Neighbourhood Theorem

Definition 6.7. Suppose $\rho \colon Z \longrightarrow X$ is a bundle. Then a map $s \colon X \longrightarrow Z$ is a *section* of ρ if $\rho \circ s = \mathrm{Id}_X$.

Theorem 6.1. *Suppose $F^{n-k} \subset M^n$ is properly embedded. Then there is a closed neighbourhood $N = N(F) \subset M$ of F and a $\mathbb{B}^k$–bundle map such that*

1. *the inclusion $i \colon F \longrightarrow N(F)$ is the zero section, i.e. $i(x) = 0 \in \mathbb{B}^k = \rho^{-1}(x)$,*
2. *N is a codimension 0 submanifold of M (with corners) and*

3. any $N'(F)$ satisfying the properties (1) and (2) is ambient isotopic to $N(F)$ fixing F pointwise.

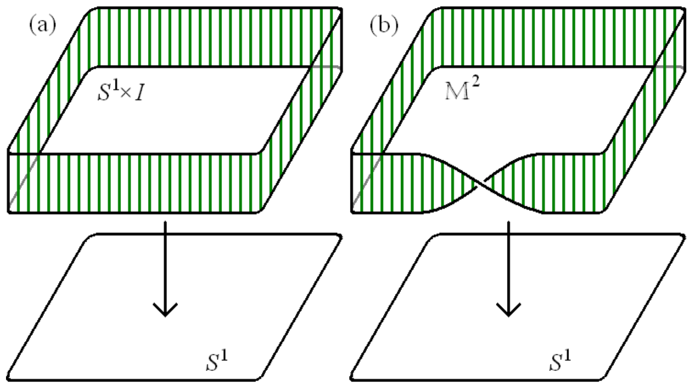

Figure 9: Two inequivalent bundles over S^1: (a) $\mathbb{B}^1 \times S^1$ and (b) $\mathbb{B}^1 \widetilde{\times} S^1$.

Notation: We denote by $n(F)$ the interior of $N(F)$. Furthermore, M *cut along* F, is the manifold (perhaps with corners) $M - n(F)$. When F is codimension 1 manifold there is a regluing map $M - n(F) \stackrel{\text{reglue}}{\longrightarrow} M$.

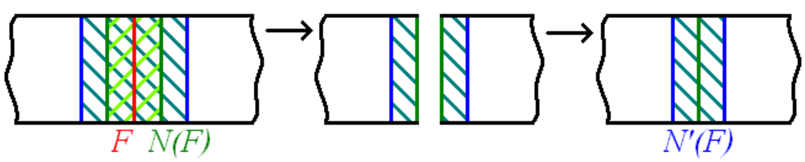

Figure 10: Cut open along $N(F)$ and glue back along $N'(F)$.

Exercise 6.8. All I–bundles over S^2 are trivial.

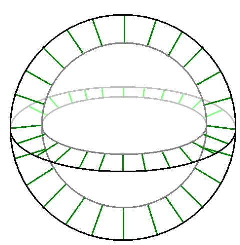

Figure 11: The trivial I–bundle over S^2.