MA4J2 Three Manifolds

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Lecture 7

Suppose that $\rho\colon G^2 \to F^2$ is a double cover. Roughly, this corresponds to an index two subgroup of $\pi_1(F)$, and hence to a homomorphism $\pi_1(F) \to \mathbb{Z}/2\mathbb{Z}$. Then for all $x \in F$, $|\rho^{-1}(x)| = 2$, so there is a canonical involution $\tau\colon G \to G$, where $\tau(y)$ is defined to be the unique element of $\rho^{-1}(\rho(y)) - \{y\}$. For an example, see Figure 1.

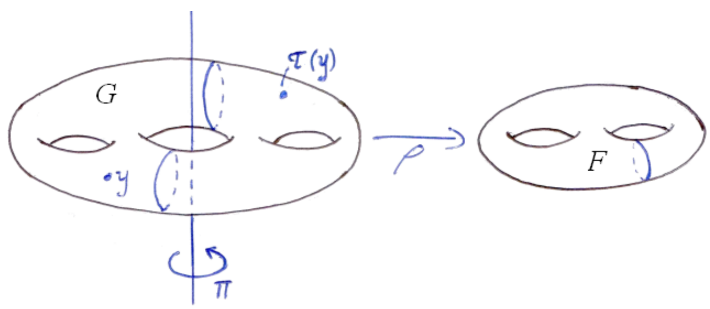

Figure 1: Here the involution τ is rotation by π about an axis.

Define $T = (G \times I)/\sim$, where $(y, 0) \sim (\tau(y), 0)$. Then $P\colon T \to F$ given by $(y,t) \mapsto \rho(y)$ is an I–bundle over F. Now suppose that $\rho\colon G \to F$ is the orientation double cover; so $G = F \times \{0,1\}$ if F is orientable, and G is orientable if F is not; for example $\mathbb{T}^2 \xrightarrow{\times 2} \mathbb{K}^2$ (Figure 2).

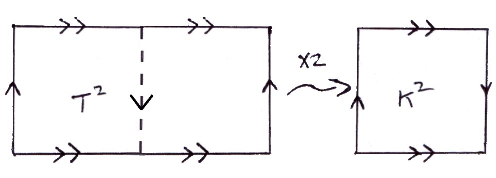

Figure 2: The torus is a double cover for the Klein bottle.

Then $P\colon T \to F$ as above is called the *orientation I–bundle* (Figure 3).

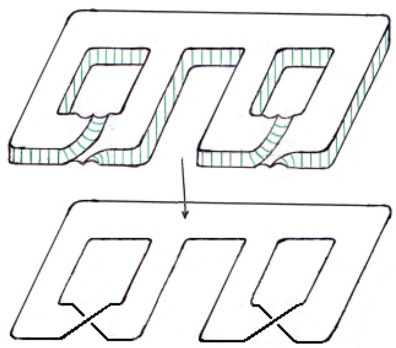

Figure 3: The orientation I–bundle over $\mathbb{K}^2 - \operatorname{int}\left(\mathbb{D}^2\right)$.

We have the following:

Theorem 7.1. *Suppose that $(F^2, \partial M) \subset (M^3, \partial M)$ is properly embedded. Then $N(F)$ is bundle equivalent to an I–bundle over F. If additionally M is orientable, then $N(F)$ is bundle equivalent to the orientation I–bundle over F.*

Example 7.1. Figure 4 shows the I–bundle for $\mathbb{T}^2$.

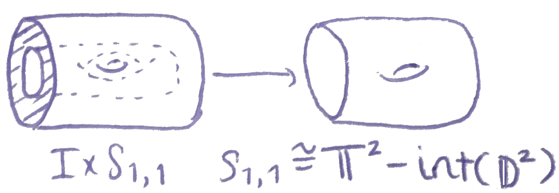

Figure 4: The orientation I–bundles are the only I–bundles one can draw in three-space.

Definition 7.1. We say that $F \subset M$ is *two-sided* if F separates $N(F)$. Otherwise F is *one-sided*.

Example 7.2. The core curve α in the Möbius band $\mathbb{M}^2$ is one-sided. $\mathbb{D}^2 \times \{p\} \subset \mathbb{D}^2 \times S^1$ is two-sided for any $p \in S^1$. We can also find a Möbius band in $\mathbb{D}^2 \times S^1$ that is one-sided. $\mathbb{M}^2 \times \left\{\frac{1}{2}\right\}$ is two-sided in $\mathbb{M}^2 \times I$; see Figure 5.

Exercise 7.1. If $F \subset M$ is properly embedded, give a relationship between the orientability of M and F, and the number of sides of F.

Definition 7.2. If $\rho\colon T \to F$ is an I–bundle, then $X \subset T$ is *vertical* if X is a union of fibres.

Definition 7.3. The *vertical* boundary of an I–bundle $\rho\colon T \to F$ is $\partial_v T := \rho^{-1}(\partial F)$.

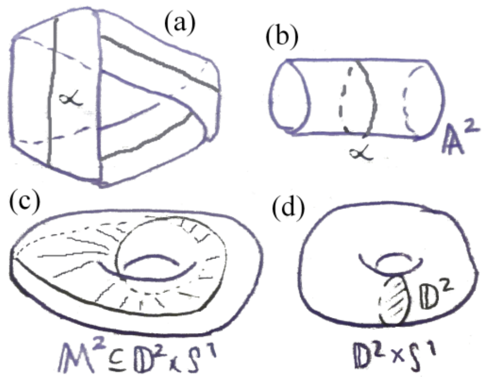

Figure 5: (a) α is one-sided in $\mathbb{M}^2$. (b) α is two-sided in $\mathbb{A}^2$ (c) $\mathbb{M}^2$ is one-sided in $\mathbb{D}^2 \times S^1$ (d) $\mathbb{D}^2$ is two-sided in $\mathbb{D}^2 \times S^1$.

Definition 7.4. The *horizontal* boundary of an I–bundle $\rho\colon T \to V$ is $\partial_h T = \partial T - \operatorname{int}(\partial_v T)$.

Exercise 7.2. $\partial_v T$, $\partial_h T$ and the zero section are all incompressible in T, except for $\partial_v T$ when $T = I \times \mathbb{D}^2$.

Exercise 7.3. If $\partial F \neq \varnothing$, F is compact and connected, and $\rho\colon T \to F$ is the orientation I–bundle, then T is a handlebody.

Before moving on, we summarize examples of 3–manifolds discussed so far.

Example 7.3. We have seen:

(i) S^3, $\mathbb{P}^3$ and $\mathbb{T}^3$, which are closed.

(ii) V_g, the handlebodies.

(iii) I–bundles and S^1–bundles over surfaces.

8 Lecture 8: Triangulations

Definition 9.1. Define the *k–simplex* by:

$$\Delta^k = \left\{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : \sum x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i\right\}$$

Definition 9.2. The *facet* $\delta_I \subset \Delta^k$ is the subsimplex of the form:

$$\delta_I = \{(x_0, ..., x_k) \in \Delta^k : x_i = 0 \text{ for all } i \in I\}$$

Definition 9.3. If $\delta \subset \Delta$ and $\delta' \subset \Delta'$ are *faces* (codimension 1 facets), then a *face pairing* is an isometry $\varphi\colon \delta \to \delta'$.

Definition 9.4. We call a collection T of simplices and face pairings a *triangulation*.

Remark. We require that for every face pairing $\varphi \in T$ that if $\varphi \colon \delta \to \delta'$ then $\delta \neq \delta'$.

Definition 9.5. The number of simplices is written $|T|$. The *underlying space* is written $||T||$, and is defined by:

$$||T|| := \left(\bigsqcup \Delta_i\right) / \{\varphi_j\}$$

Definition 9.6. The quotient map is given by $\pi \colon \bigsqcup \Delta_i \to ||T||$ and we define $\pi_i \colon \Delta_i \to ||T||$ by restriction: $\pi_i = \pi|\Delta_i$.

Example 9.1. If T is the pair of simplices in Figure 6 with face pairings given by the arrows, then $||T|| \cong \mathbb{T}^2$.

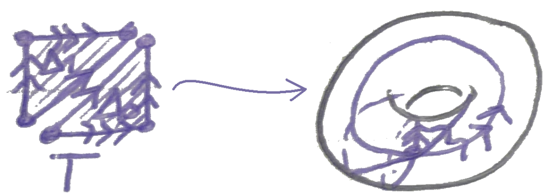

Figure 6: $||T|| \cong \mathbb{T}^2$.

Similarly, if we draw T as in Figure 7 then $||T|| = \mathbb{M}^2$.

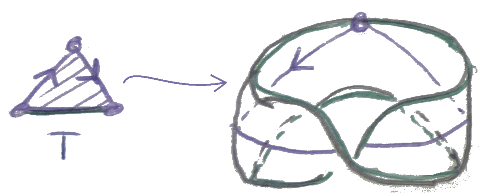

Figure 7: $||T|| \cong \mathbb{M}^2$.

Exercise 9.1. Find necessary and sufficient combinatorial conditions on T so that $||T||$ is a (PL) manifold of dimension 1, 2 or 3.

Hauptvermutung (Moise)**.** *Every topological* 3*–manifold admits a triangulation, unique up to subdivision. In particular, for any* M^3*, there exists a triangulation* T *such that* $||T|| \cong M$*.*

Remark. This is one important step in showing, in dimension three, that the categories TOP, PL and DIFF are all equivalent.

Definition 9.7. Suppose (M^3, T) is a triangulated manifold. An *orientation* of M is a choice of orientation for all $\Delta \in T$, such that all face pairings reverse the induced orientation on faces.

Example 9.2. The annulus is orientable, but the Möbius band is not. See Figure 8.

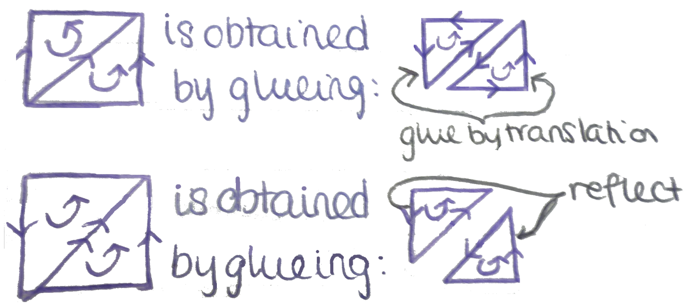

Figure 8: The annulus is orientable as all face pairings reverse the induced orientation on faces.

Proposition 9.1 (Proposition 6.5 in Lackenby)**.** *An n–manifold (M^n, T) is orientable if and only if for every simple closed curve $\alpha \in M$ we have $N(\alpha) \cong \mathbb{B}^{n-1} \times S^1$.*

Remark. We can also determine orientability in DIFF using $\text{sign}(\det(Dh))$ where h ranges over the overlap maps, as in Figure 9. We can also define orientation in TOP using homology.

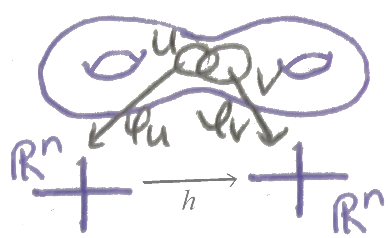

Figure 9: Orientation in DIFF arises from overlap maps of charts.

Definition 9.8. Define $\Delta^{(k)}$ to be the union of k–dimensional facets of Δ. If (M, T) is a triangulated 3–manifold, define $M^{(k)}$, the *k–skeleton* of M to be the manifold with triangulation $T = \bigcup_{i=1}^{|T|} \pi_i(\Delta^{(k)})$. Figure 10 shows the k-skeleta of Δ.

Example 9.3. Figure 11 shows two examples of identifications.

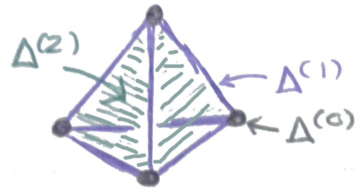

Figure 10: The k–skeleta of Δ.

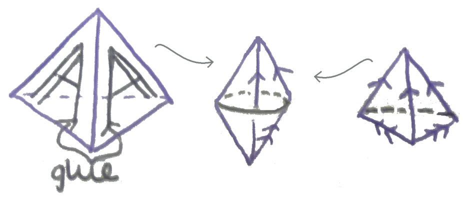

Figure 11: Two different views of the same triangulation for $\mathbb{B}^3$.

Exercise 9.2. Verify that the triangulation in Figure 12 is a three-manifold, and recognise it.

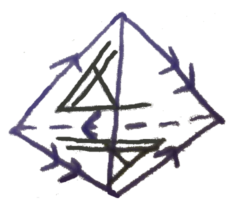

Figure 12: Which three-manifold is this?

Definition 9.9. An isotopy $F\colon M \times I \to M$ is *normal* with respect to a triangulation T of M if for all $t \in I$, the homeomorphism F_t preserves $M^{(k)}$ for all k, and $F_0 = \mathrm{Id}_M$. See Figure 13 for an example.

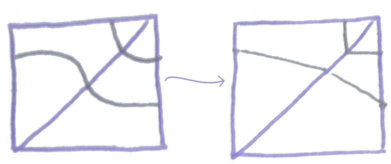

Figure 13: A normal isotopy.

Remark. Thus $M^{(0)}$ is fixed pointwise, and all other facets are fixed setwise.

Definition 9.10. Say an arc $(\alpha, \partial\alpha) \subset (\Delta^2, \partial\Delta)$ is *normal* if the points of $\partial\alpha$

are in distinct edges of Δ, and $\alpha \cap \Delta^{(0)} = \varnothing$. See Figure 14 for some examples and a non-example.

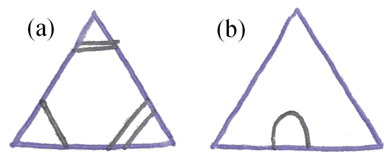

Figure 14: (a) Normal arcs. (b) This is not a normal arc.

Definition 9.11. A disk $(D, \partial D) \subset (\Delta^3, \partial\Delta)$ is a *normal disk* if ∂D is transverse to $\Delta^{(1)}$, ∂D meets each edge of $\Delta^{(1)}$ at most once, and $D \cap \Delta^{(0)} = \varnothing$. See Figures 15(a) and (b) for examples and 15(c) and (d) for non-examples.

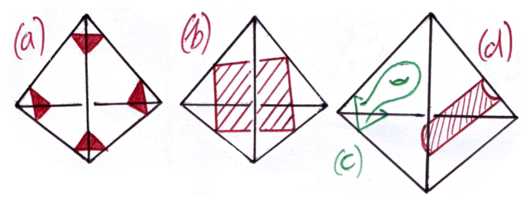

Figure 15: (a) There are four normal triangles. (b) There are three normal quadrilaterals. (c) This is not even a disk, let alone normal. (d) This is also not a normal disc.

Exercise 9.3. Prove that:

(i) There are only three normal arcs up to normal isotopy.

(ii) There are only seven normal disks up to normal isotopy.

Recall that $\pi_i \colon \Delta_i \to M$ is defined by $\pi_i = \pi|\Delta_i$, where π is the quotient map.

Definition 9.12. Suppose $S \subset M$ is a surface. Say S is normal if $\pi_i^{-1}(S)$ is a disjoint collection of *normal* disks for all i.

Example 9.4. The three normal disks in the tetrahedron shown in Figure 16 give a normal surface under the identification indicated by the arrows.

Exercise 9.4. Show that, with triangulations as in Figure 17, (a) and (b) are three manifolds, and recognise them.

Theorem 9.2 (Haken-Kneser Finiteness). *Suppose (M, T) is a connected, compact triangulated 3–manifold. Suppose $S \subset (M, T)$ is an embedded normal surface. Then if $|S| \geq 20|T| + 1$ there are components $R, R' \subset S$ so that R, R' cobound a product component of $M - S$.*

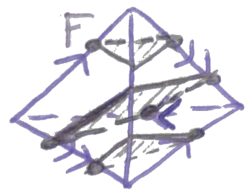

Figure 16: Recognise the normal surface F by computing $|\partial F|$, $\chi(F)$ and the orientability.

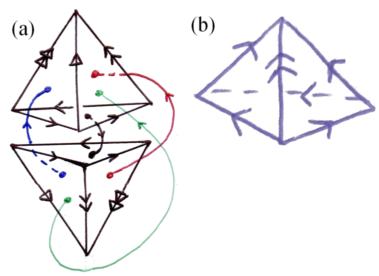

Figure 17: Show that (a) and (b) are three manifolds and recognise them.

Remark. Figures 18 and 19 show examples of parallel surfaces.

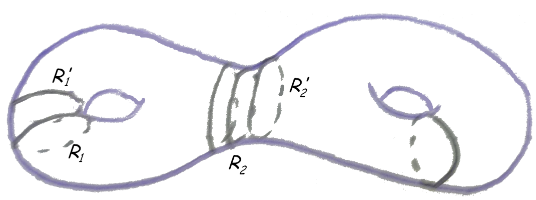

Figure 18: Here both R_1 & R_1' and R_2 & R_2' bound copies of $\mathrm{D}^2 \times I$.

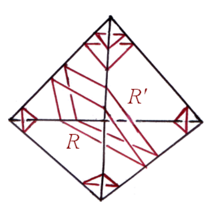

Figure 19: R and R' bound a product.

Proof of Theorem 9.2. Recall that $S \cap \Delta$ for $\Delta \in T$ is a finite collection of normal disks. Consider the subcollection of disks of a fixed *type*, that is a normal isotopy class. Call the outermost disks *ugly*, the second outermost disks *bad*, and all other disks *good*, as illustrated in Figure 20.

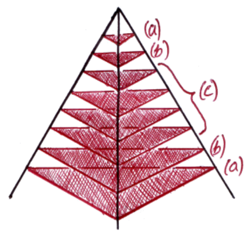

Figure 20: (a) Ugly disks. (b) Bad disks. (c) Good disks.

Thus there is a component $F \subset S$, such that F is a union of good disks. To see this, note that there are at most $20|T|$ ugly and bad disks in total. There are at most five types of disk in each $S \cap \Delta$, and at most four of each can be ugly or bad; see Figures 21(a) and (b).

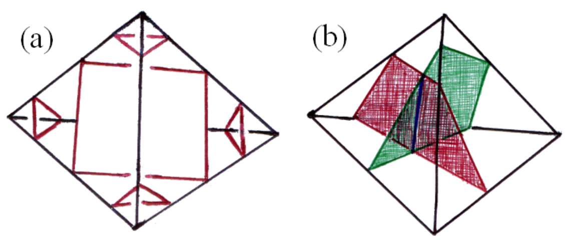

Figure 21: (a) There are at most five types of disk in each $S \cap \Delta$ because (b) two normal quadrilaterals of different types must intersect.

Now let N be the closure of the union, over all Δ_i, of all components of $\Delta_i - S$ that are adjacent to F, as in Figure 22.

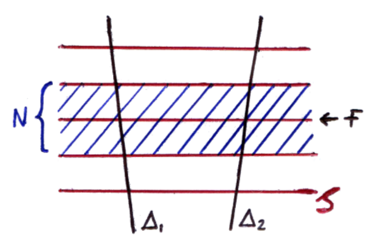

Figure 22: N is the closure of the union over all Δ_i of all components of $\Delta_i - S$ that are adjacent to F.

Exercise 9.5. Prove that N is an I–bundle and either N is ambient isotopic to $N(F)$ *or* F is two-sided and parallel to $\partial_h N$.