MA4J2 Three Manifolds

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Lecture 10

We recall properties of π_1:

Definition 10.1. Suppose A and B are groups. Then if $A = \langle a_i \mid r_k \rangle$ and $B = \langle b_j \mid s_l \rangle$, their free product $A * B$ is given by

$$A * B = \langle a_i, b_j \mid r_k, s_l \rangle .$$

Theorem 10.1 (van Kampen)**.** *If $W = X \cup_Z Y$ and Z is path connected (as in Figure 1), then, choosing a base point $p \in Z$, $\pi_1(W,p) \cong \pi_1(X,p) * \pi_1(Y,p)/N$, where N is the normal subgroup generated by:*

$$\{i_*(z)(j_*(z))^{-1} : z \in \pi_1(Z,p)\}$$

where $i : Z \hookrightarrow X$ and $j : Z \hookrightarrow Y$ are the inclusions.

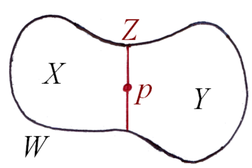

Figure 1: If $W = X \cup_Z Y$, then $\pi_1(W) = (\pi_1(X) * \pi_1(Y))/N$.

Corollary 10.1. *If $\pi_1(Y,p) = \{\mathbf{1}\}$ then $\pi_1(W,p) = \pi_1(X,p)/N$ where N is the normal subgroup generated by:*

$$\{i_*(z) : z \in \pi_1(Z,p)\}.$$

Corollary 10.2. *If $\pi_1(Z,p) = \{\mathbf{1}\}$ then $\pi_1(W,p) = \pi_1(X,p) * \pi_1(Y,p)$.*

Proposition 10.3. *If (M,T) is triangulated then $\pi_1(M) = \pi_1\left(T^{(2)}\right)$.*

Exercise 10.1. Prove Proposition 10.3. See Figure 2 for a hint.

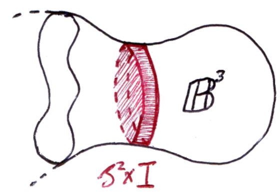

Figure 2: Hint: Attach 3-balls one by one.

Proposition 10.4. *$\pi_1(T^{(2)}) = \pi_1(T^{(1)})/N$ where N is the normal subgroup generated by boundaries of two-simplices in T. Note that $\pi_1(T^{(1)})$ is a free group, as $T^{(1)}$ is a connected graph.*

We now give several example computations.

Example 10.1. Consider Figure 3, where the faces are glued according to the arrows.

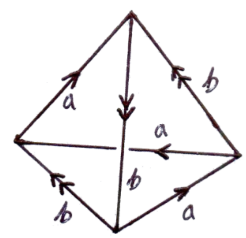

Figure 3: What is the fundamental group of this manifold?

Exercise 10.2. Check that this is a 3–manifold.

Step 1: Find a spanning tree for $T^{(1)}$. Here $T^{(1)}$ is the graph shown in Figure 4 and so the spanning tree is just the vertex.

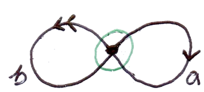

Figure 4: $T^{(1)}$ in this case. The spanning tree is the single vertex circled in green.

Step 2: Give labels to the non-tree edges of $T^{(1)}$, as in Figure 4.

Step 3: Read off relations from faces of $T^{(2)}$. There is one relation per face in the quotient. Here we have $\langle a, b \mid a^2 = b,\ b^2 a = \mathbf{1} \rangle$.

Step 4: (optional) Use Tietze transformations to simplify:

$$\left\langle a, b \mid a^2 = b,\ b^2 a = \mathbf{1} \right\rangle \cong \left\langle a \mid (a^2)^2 a = \mathbf{1} \right\rangle \cong \mathbb{Z}/5\mathbb{Z}.$$

Example 10.2. (A non-Abelian example.) The *one-quarter turn space* Q is the quotient of the unit cube as shown in Figure 5:

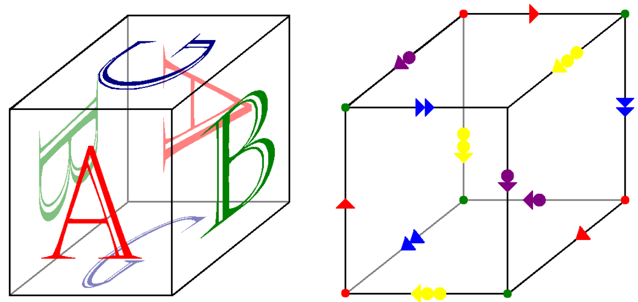

Figure 5: Two visualisations of how to glue faces to get Q.

Step 1: The 1–skeleton is the graph in Figure 6(a) with four edges and two vertices. We take the circled edge as the spanning tree.

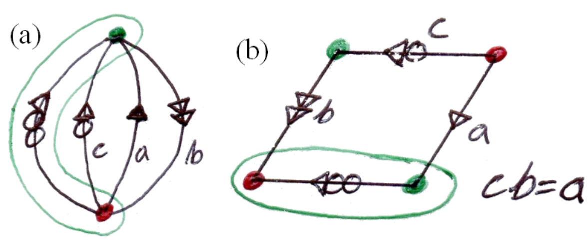

Figure 6: (a) The 1–skeleton and spanning tree. (b) After labelling the non-tree edges, read off relators from the faces. Edges of the spanning tree do not contribute to the relators.

Step 2: Label the non-tree edges with a, b, c.

Step 3: The three squares give relations and we have the following presentation

$$\pi_1(Q) = \langle a, b, c \mid a = cb,\ ba = c,\ abc = \mathbf{1} \rangle\,.$$

Exercise 10.3. Recognize $\pi_1(Q)$. In particular, it is not Abelian.

Abelian groups

Definition 10.2. Suppose that Z is an Abelian group. Define $N := \{z \in Z : z$ is finite order$\}$. Then $N < Z$ is called the *torsion subgroup* of Z.

Recall that $A \oplus B$ is the *direct product* of A and B.

Proposition 10.5. *Suppose Z is a finitely generated Abelian group. Then there exist unique $k \in \mathbb{N}$ and N a finite group so that $Z \cong \mathbb{Z}^k \oplus N$.*

Proof. This follows from the classification of finitely generated Abelian groups. □

Definition 10.3. We call k the *rank* of Z, and use the notation $\mathrm{rk}(Z) = k$.

Definition 10.4. Let G be any (finitely generated) group. The *commutator subgroup* of G is $[G, G]$, the subgroup of G generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.

$$[G, G] = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle \lhd G.$$

Definition 10.5. We define the *Abelianization* of G to be $G^{\mathrm{Ab}} = G/[G, G]$.

Definition 10.6. We define the *first homology group* of M^3 to be $H_1(M, \mathbb{Z}) := [\pi_1(M)]^{\mathrm{Ab}}$.

Example 10.3. Let $M^3 = N^3 \# P^3$. Then it follows by van Kampen's theorem that $\pi_1(M) \cong \pi_1(N) * \pi_1(P)$. Therefore $H_1(M) = H_1(N) \oplus H_1(P)$.

Exercise 10.4. Show that $(A * B)^{\mathrm{Ab}} = A^{ab} \oplus B^{\mathrm{Ab}}$.

Example 10.4. As in the last example we have that

$$\pi_1(\#_g S^2 \times S^1) = F_g \cong *_g\mathbb{Z},$$

so $H_1(\#_g S^2 \times S^1) = \mathbb{Z}^g$ has rank g. We denote $\#_g S^2 \times S^1$ by M_g.

Proposition 10.6. *If M is connected, orientable, compact and $M \cong N \# M_g$, then $g \leq \mathrm{rk}(H_1(M))$.*

Note here that π_1 is finitely generated since M is compact.

Proof. We know that $H_1(M) = H_1(N) \oplus H_1(M_g)$, so:

$$\mathrm{rk}(H_1(M)) = \mathrm{rk}(H_1(N)) + g.$$ □

This is the first step in the existence proof for connect sum decompositions. For the next step, we need the following proposition:

Proposition 10.7. *Suppose M is connected, orientable and compact. Then there exists a decomposition*

$$M \cong \#_{i=1}^k N_i \# (\#_g S^2 \times S^1) \# (\#_n \mathbb{B}^3)$$

where each N_i is irreducible and not S^3, $\mathbb{B}^3$ or $S^2 \times S^1$.

Proof.

Step 1: Let n be the number of components of ∂M that are 2–spheres. Let F be the frontier of a "tree-like" union of arcs and two-sphere boundary components, as shown in Figure 7. Form $M - n(F)$ and cap off $F^{\pm}$ by 3–balls. From now on we assume that $n = 0$.

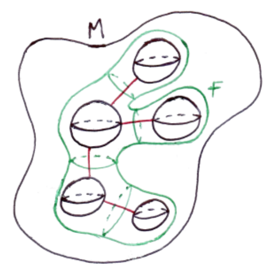

Figure 7: F is the frontier of a "tree-like" union of arcs and two-sphere boundary components.

Step 2: Proposition 10.6 gives us an upper bound on the number of summands of M homeomorphic to $S^2 \times S^1$. Thus from now on we may assume that $g = 0$. It follows that any 2–sphere embedded in M separates.

For Step 3, we require the following definitions.

Definition 10.7. We define $S^3_k := \#_{i=1}^k \mathbb{B}^3$ and we call this a *ball with holes* or a *punctured sphere*. See Figure 8.

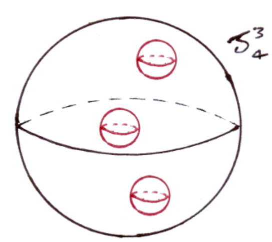

Figure 8: Here, $n = 4$.

Exercise 10.5. Show that $(\#_n \mathbb{B}^3) \cup_{S^2} (\#_m \mathbb{B}^3) \cong \#_{n+m-2} \mathbb{B}^3$.

Definition 10.8. We call $S \hookrightarrow M$ a *sphere system* if S is an embedding of a disjoint collection of 2–spheres; see Figure 9.

Definition 10.9. A system $S \hookrightarrow M$ is *reduced* if no component of $M - n(S)$ is homeomorphic to a punctured sphere. The sphere system in Figure 9 is reduced.

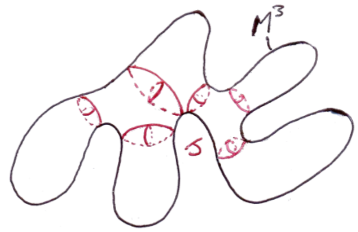

Figure 9: A reduced sphere system S in M.

Lectures 11 and 12

Step 3: If M is irreducible we are done. If $M \cong S^3$ we are done by Alexander's theorem. So suppose that M contains an essential 2–sphere. For the reminder of the proof, we fix a finite triangulation T of M. So our assumptions give us a reduced sphere system $S \subset M$.

Normalization Lemma. For any reduced sphere system $S \subset M$ there is a normal, reduced sphere system S' such that $|S'| \geq |S|$.

If we assume this lemma, we get the following proposition:

Proposition 11.8. *(Existence) Let M be defined as above. Then $M \cong \#_{i=1}^{n} N_i$ such that all N_i are irreducible and $N_i \ncong S^3, \mathbb{B}^3$.*

Proof. Let S_1 denote an essential 2–sphere, so $M = N_1 \#_{S_1} N_2$. If N_1 is homeomorphic to $\#_k \mathbb{B}^3$ for $k \geq 1$, then we have a contradiction. So, $S = \{S_1\}$ is a reduced sphere system. Let $\overline{S}$ be a maximal sphere system (i.e. of maximal size). This exists because any normal reduced system has at most $20|T|$ components; this follows from the Haken-Kneser finiteness and the normalization lemma. Since $\overline{S}$ is maximal , if we cut M along $\overline{S}$ and cap off with 3–balls the resulting manifolds $\{N_i\}$ are all irreducible. □

To prove the normalization lemma, we must *normalize* the given system S.

Proof of Normalization Lemma. Isotope S to be transverse to $T^{(k)}$ for $k = 0, 1, 2$, i.e. $S \cap T^{(0)} = \varnothing$, $|S \cap T^{(1)}| =: w(S)$ (the *weight* of S) is finite, $S \cap T^{(1)}$ is transverse and $S \cap \partial\Delta_i$ is a finite collection of simple closed curves; see Figure 10. We alternatingly apply *surgery* and the *baseball move.*

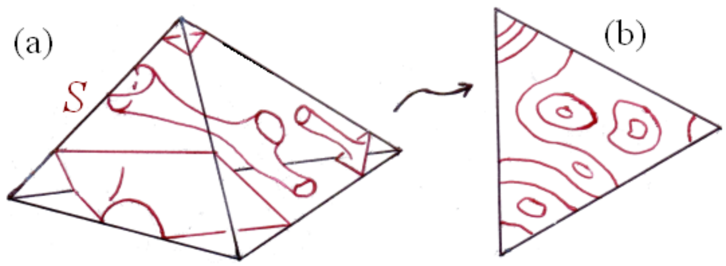

Figure 10: (a) The sphere system can look unpleasant in the triangulation. (b) A possible picture of $S \cap T^{(2)}$.

Surgery: Suppose $(D, \partial D) \subset (M, S)$ is a surgery disk, i.e. $D \cap S = \partial D$. Suppose $D \cap S \subset F$ is a component of S. As before, define $F_D = F - n(D) \cup D^+ \cup D^-$. Define $S_D = (S - F) \cup F_D$. Notice that ∂D separates F, so $F_D = F^+ \cup F^-$. See Figure 11.

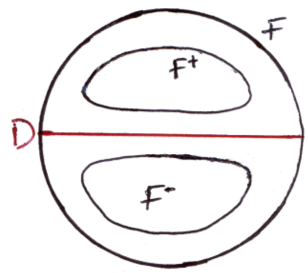

Figure 11: Notice that ∂D separates F, so $F_D = F^+ \cup F^-$.

Let $X, Y \subset M - n(S)$ be the components adjacent to F and suppose $D \cap X \neq \emptyset$. So let $X^+ \cup X_0 \cup X^- = X - n(F_D)$ where X_0 meets D and $X^\pm$ are adjacent to $F^\pm$, respectively. See Figure 12.

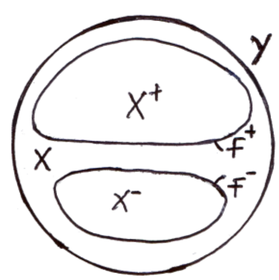

Figure 12: $X^+ \cup X_0 \cup X^- = X - n(F_D)$, where X_0 meets D, and $X^\pm$ are adjacent to $F^\pm$.

Note that $X_0 \cong \#_3\mathbb{B}^3$. Since we assumed S is a reduced sphere system, we find Y is not a punctured sphere.

Exercise 11.6. $Y \cup_F X_0$ is not a punctured sphere.

Claim. At most one of X^+, X^- is a punctured sphere.

Proof. If both are punctured spheres then so is $X = X^+ \cup_{F^+} X_0 \cup_{F^-} X^-$, a contradiction. This proves the claim. ☐

Let $S' = S - F$ thus either $S^+ = S' \cup F^+$ or $S^- = S' \cup F^-$ or $S_D = S' \cup F_D$ is a reduced system.

Using surgery: For every tetrahedron $\Delta \in T^{(3)}$, the surface S meets $\partial\Delta$ is a collection of simple closed curves. See Figure 13 for a possible intersection pattern.

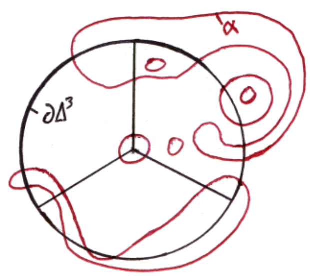

Figure 13: A possible intersection of S with the boundary of a tetrahedron.

For every simple closed curve $\alpha \subset \partial\Delta \cap S$ we do the following. Pick a disk $D \subset \partial\Delta$ bounded by α. Isotope D into Δ (∂D stays in S), as in Figure 14.

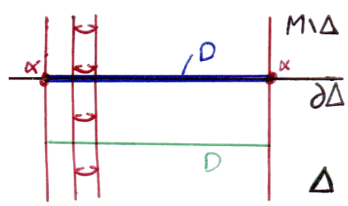

Figure 14: Isotope D into Δ.

Use D (in Δ) to surger all curves of $S \cap D$, innermost first. When this is done, $S \cap \Delta$ is a collection of disks (for all Δ).

Claim. After surgery, for all Δ and for all simple closed curves $\alpha \subset \partial\Delta \cap S$, α meets $\Delta^{(1)}$.

Proof. Suppose α has weight 0 and $\alpha \subset f \subset \Delta^{(2)}$ a face. We surgered along both $D^{\pm}$, so the component sphere containing α bounds a ball as in Figure 15.

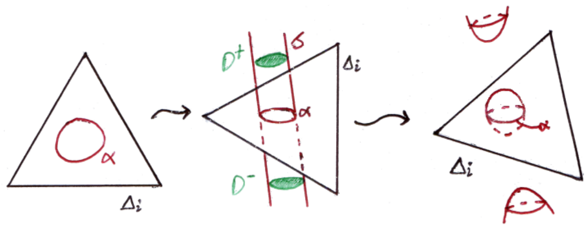

Figure 15: We surgered along both $D^{\pm}$, so the component sphere containing α bounds a ball.

But surgery deletes trivial spheres. This proves the claim. $\square$

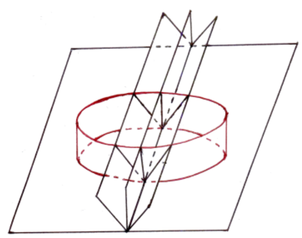

Figure 16: The intersection of S with the two-skeleton; outside of Δ it can be complicated.