MA4J2 Three Manifolds

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Lecture 13

Proof. We complete the proof of the existence of connect sum decomposition.

Procedure 2: Baseball move. We perform this move after surgery along all curves of $S \cap \partial\Delta$ for all $\Delta^3 \in T$. Suppose α is a simple closed curve of $S \cap \partial\Delta$, where $\Delta^3 \in T$. So α bounds disks D_0 and D_1 in $\partial\Delta$. Suppose that there is an edge $e \in \Delta^{(1)}$ with $|\alpha \cap e| \geq 2$, as illustrated in Figure 1.

Exercise 13.1. Without loss of generality, there is a component $d \subset D_0 \cap e$ such that $d \cap \Delta^{(0)} = \varnothing$, as in Figure 1.

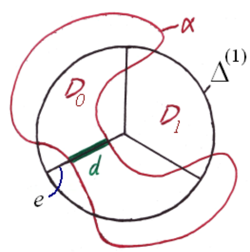

Figure 1: α bounds two disks D_0 and D_1, and there is an edge $e \in \Delta^{(1)}$ such that $|\alpha \cap e| = 2$.

Now let $D = D_0$. By an innermost arc argument we may assume that $d \cap S = \partial d$. Let $D' \subset S \cap \Delta$ be the disk bounded by α, as in Figure 2.

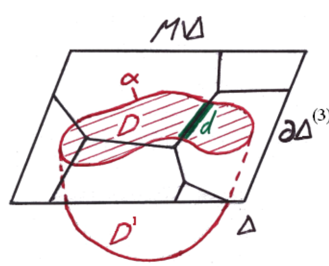

Figure 2: D' is the disk bounded by α.

Since $D \cup D' \cong S^2$, they cobound a three-ball, B, by Alexander's theorem, and so we may choose an embedded arc $d' \subset D'$ so that d and d' cobound a disk $E \subset B$, as in Figure 3.

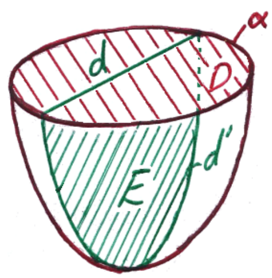

Figure 3: The arcs d and d' cobound a disk $E \subset B$.

Let C be the 3–ball obtained from $N(E)$ by cutting along S and retaining the component containing E; see figure 4.

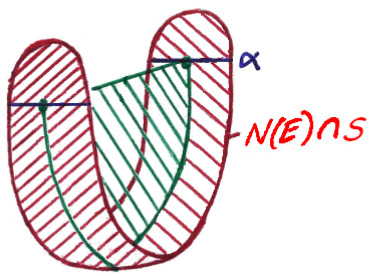

Figure 4: A picture of $N(E) \cap S$.

Write $\partial_- C = C \cap S$ and $\partial_+ C = \overline{\partial C - \partial_- C}$. The *baseball curve* is the common boundary $\partial\partial_+ C = \partial\partial_- C$, as in Figure 5.

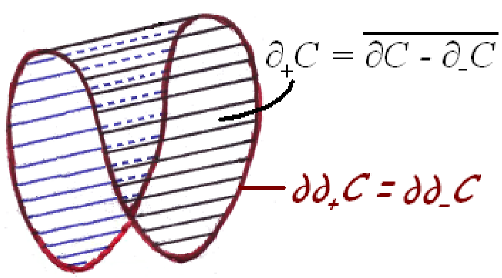

Figure 5: The baseball curve is the common boundary $\partial\partial_+ C = \partial\partial_- C$.

Since C is a 3–ball, there is an isotopy, called the *baseball move*, taking $\partial_- C$ to $\partial_+ C$; see Figures 6(a) or (b). This gives an isotopy of S to S'. Notice that $w(S') = w(S) - 2$.

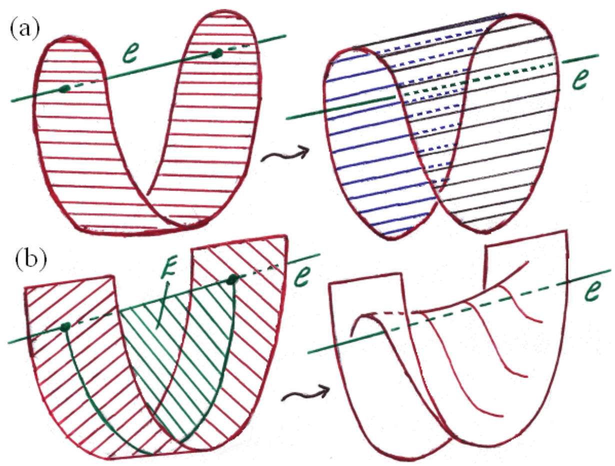

Figure 6: Two visualisations of the baseball move.

So alternate between surgery along all curves and single baseball moves. As $w(S)$ is decreasing, this process terminates with S in normal position. If $w(S) = 0$ then $S = \varnothing$ and this is a contradiction as surgery never decreases the initial number of essential spheres. So this completes the proof of existence. $\square$

Following Hatcher, for uniqueness we use lemma 13.1.

Definition 13.1. If M is a 3–manifold, define $\widehat{M}$ to be M with all $S^2 \subset \partial M$ capped off by 3–balls, and discarding 3–sphere components.

Lemma 13.1. *Suppose that $S \subset M$ is a sphere system (not necessarily reduced) so that:*

$$\widehat{M - n(S)} = \bigsqcup_{i=1}^{k} N_i$$

is a disjoint union of irreducible manifolds. Suppose that $(D, \partial D) \subset (M, S)$ is a surgery disk. Then:

$$\widehat{M - n(S_D)} = \bigsqcup_{i=1}^{k} N_i.$$

Exercise 13.2. Prove this lemma. For a hint, see Figure 7.

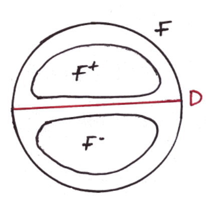

Figure 7: Hint for Exercise 13.2.

So we may now complete the proof of uniqueness of prime decomposition.

Proof of uniqueness. Suppose S and T are sphere systems so that:

$$M - n(S) = \bigsqcup_{i=1}^{k} P_i$$

and

$$N - n(T) = \bigsqcup_{j=1}^{l} Q_j$$

where the P_i and Q_j are irreducible. Now, if $S \cap T = \varnothing$ we have:

$$\begin{aligned} \bigsqcup P_i &= \widehat{\bigsqcup P_i - n(T)} \\ &= \widehat{M - n(S \cup T)} \\ &= \widehat{\bigsqcup Q_j - n(S)} = \bigsqcup Q_j \end{aligned}$$

On the other hand, if $S \cap T \neq \varnothing$ then surger S along an innermost disk of T and apply Lemma 13.1. Finally, if $M \cong N\#\big(\#_l S^2 \times S^1\big)$ and $M \cong N\#\big(\#_k S^2 \times S^1\big)$ then:

$$\operatorname{rank}(H_1(N)) + l = \operatorname{rank}(H_1(M)) = \operatorname{rank}(H_1(N)) + k$$

and so $l = k$. □

Lecture 14

Exercise 14.3. Suppose that (M, T) is orientable, compact, connected, irreducible and triangulated. Suppose $F \subset M$ is embedded, closed ($\partial F = \varnothing$, compact) and orientable. Show that if G is incompressible, it is isotopic to a normal surface.

Definition 14.2. Say F properly embedded in M is *boundary parallel* if there is an isotopy (relative to ∂F) pusing F into ∂M. More precisely, there is an isotopy $H\colon F \times I \to M$ such that:

(i) H_t is an embedding of F into M for all $t < 1$.

(ii) H_1 is an embedding of F into ∂M.

(iii) $H_0 = \text{Id}$.

(iv) $H_t|\partial F = \text{Id}$.

Equivalently $M - n(F)$ has a component $X \cong F \times I$ with $F \times \{0\} = F^+ \subset N(F)$ and $F \times \{1\} \subseteq \partial M$. See Figure 8.

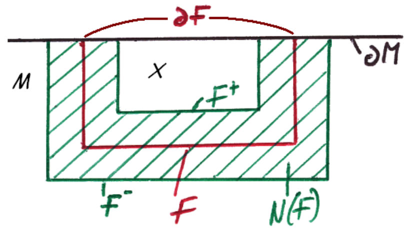

Figure 8: F is boundary parallel to M.

Example 14.1. (See Figure 9)

(i) The equatorial disk $\mathbb{B}^2 \subset \mathbb{B}^3$ is boundary parallel.

(ii) Take $K \subset T = \partial(\mathbb{D}^2 \times S^1)$. Let $N(K)$ be a closed neighbourhood in $\mathbb{D}^2 \times S^1$. Let $G = N(K) \cap T$. So $G \subset T = \partial(\mathbb{D}^2 \times S^1)$. Let $F = \overline{\partial N(K) - G}$, so F is boundary parallel; in fact parallel to G.

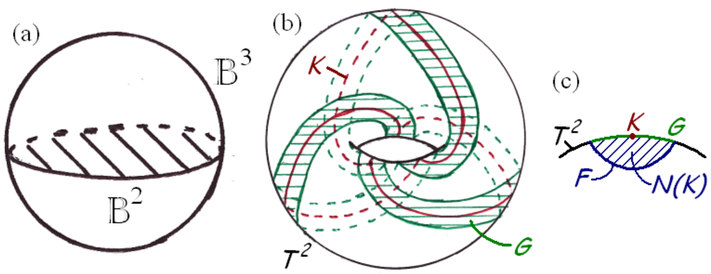

Figure 9: (a) Example (i). (b) Example (ii). (c) Cross section for Example (ii).

Note. F in example (ii) above is boundary parallel in essentially a unique way, unlike $\mathbb{B}^2 \subset \mathbb{B}^3$, or the following. Take $\mathbb{B}^1 \times S^1 \subseteq \mathbb{D}^2 \times S^1$. Then this is boundary parallel in two ways; see Figure 10.

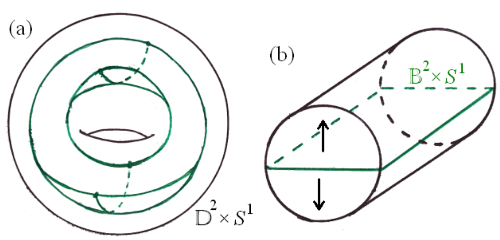

Figure 10: (b) is a cross section of (a), and $\mathbb{B}^1 \times S^1$ can be isotoped either up or down into $\mathbb{T}^2 = \partial\left(\mathbb{D}^2 \times S^1\right)$.

Example 14.2. $\mathbb{M}^2 \subseteq \mathbb{D}^2 \times S^1$ is not boundary parallel; see Figure 11.

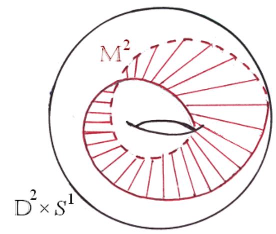

Figure 11: $\mathbb{M}^2$ is not boundary parallel in $\mathbb{D}^2 \times S^1$.

Definition 14.3. A torus $T \subset M$ is essential if it is incompressible and not boundary parallel.

Definition 14.4. Suppose M is irreducible, orientable, compact and connected. Then the manifold M is *toroidal* if there exists an essential torus $T \subset M$. M is *atoroidal* if there are no essential tori embedded in M.

Example 14.3. Suppose $K \subset S^3$ is a knot. Define the *knot exterior* $X_K := S^3 - n(K)$. If $K = L \# L'$ is a non-trivial connect sum of knots, then X_K is toroidal.

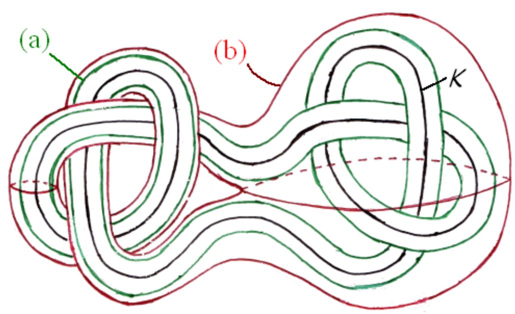

Figure 12: (a) $n(K)$. (b) An essential torus in X_K.

As shown in the previous lecture, when dealing with essential 2–spheres, we cut and cap off with 3–balls. However, there is no canonical way to cap off $\mathbb{T}^2 \subset \partial M$. So we must live with the possibility of incompressible tori, but at least we may eliminate essential tori.

Definition 14.5. Fix K, a knot in S^3, called the *companion knot.* Fix $L \subset \mathbb{D}^2 \times S^1$, the *pattern knot.* Fix a homeomorphism $\varphi \colon \mathbb{D}^2 \times S^1 \to N(K)$. Then $\varphi(L) \subset S^3$ is a *satellite knot* with pattern L and companion K. See Figure 13.

Example 14.4. All non-trivial connect sums are satellite knots.

Remark. If K is not the unknot and $L \subset \mathbb{D}^2 \times S^1$ is *disk busting* (for all compressing disks $D \subset \mathbb{D}^2 \times S^1$, $|L \cap D| \geq 1$, and L is not isotopic to $\{0\} \times S^1$), then $X_{\varphi(L)}$ is toroidal.

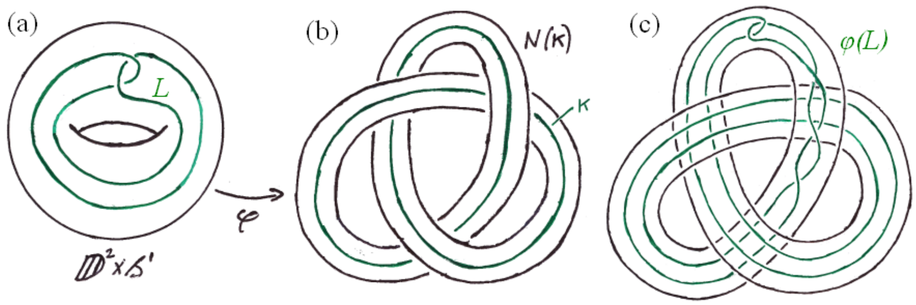

Figure 13: (a) L is the pattern knot, (b) K is the companion knot and (c) $\varphi(L)$ is the satelite knot.

Theorem 14.2 (Thurston)**.** *Every knot $K \subset S^3$ other than the unknot is either a satellite knot, a torus knot or a hyperbolic knot, as respectively X_K is toroidal, X_K is atoroidal but cylindrical, or X_K is atoroidal and acylindrical.*

Exercise 14.4. Show that X_K is irreducible.

Example 14.5. S^3 is atoroidal, but $\mathbb{T}^3$ is not; see Figure 14.

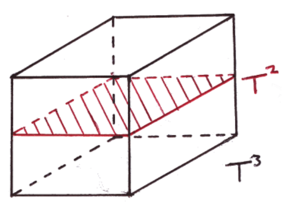

Figure 14: $\mathbb{T}^3$ contains $\mathbb{T}^2$ as an essential torus, and so is toroidal.

Lecture 15

Exercise 15.5. Suppose $F \subset M$ is properly embedded and suppose that $i_* \colon \pi_1(F) \to \pi_1(M)$ is injective. Show that F is incompressible (i.e., all surgery disks are trivial).

The final part of the course will be devoted to proving a partial converse to Exercise 15.5, via the loop theorem, the disk theorem and Dehn's lemma. An application of this converse will give us the following example:

Example 15.6. A knot $K \subset S^3$ is isotopic to a round circle (that is K is unknotted) if and only if $\pi_1(X_K) \cong \mathbb{Z}$.

Definition 15.6. A *torus system* is a finite union of disjoint, non-parallel, essential tori.

Proposition 15.3 (Corollary 1.8 in Hatcher)**.** *Suppose that M is compact, connected, orientable and irreducible. Then there is a torus system $S \subset M$ (where we allow $S = \varnothing$), so that all components of $M - n(S)$ are atoroidal.*

Proof. If M is atoroidal then take $S = \varnothing$. Otherwise, fix a triangulation T of M and suppose that $F \subset M$ is an essential torus. So $S = \{F\}$ is a torus system. We now induct on $|S|$. By Exercise 14.3 we may normalize S. By Haken-Kneser finiteness we find that $|S| \leq 20|T|$, so if there exists a component $N \subseteq M - n(S)$ which is toroidal then we find $F' \subset N$ an essential torus. So F' is not parallel to any component of S. Let $S' = S \cup \{F'\}$. Then S' is again a torus system. □

Remark. The final step uses Exercise 4.5 in Exercise Sheet 4.

Example 15.7. Suppose $\varphi\colon F \to F$ is a homeomorphism of a surface F. Define $M_\varphi = F \times I/(x, 1) \sim (\varphi(x), 0)$. Then M_φ is a surface bundle over S^1 via $\rho\colon M_\varphi \to S^1$, where $\rho\colon (x,t) \mapsto t \in \mathbb{R}/\mathbb{Z}$; see Figure 15.

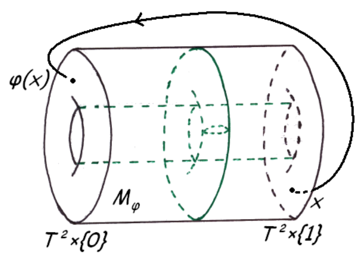

Figure 15: M_φ is a $\mathbb{T}^2$–bundle over S^1.

Exercise 15.6. Show that every fibre $T_t = \rho^{-1}(t)$ is incompressible (in fact π_1–injective) in M_φ.

Note. If $F = T \cong \mathbb{T}^2$, and $T \subset M_\varphi$ is a fibre, then $M_\varphi - n(T) \cong T \times I$. So we cannot avoid sometimes having a product component after cutting.

Remark. We have that $\mathbb{T}^3$ is the torus bundle M_{Id} in the above notation.

We now discuss lens spaces. Take $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 2\}$. Let y be the loop $\{|w| = 2\}$ and x be the loop $\{|z| = 2\}$, oriented as shown in Figure 16.

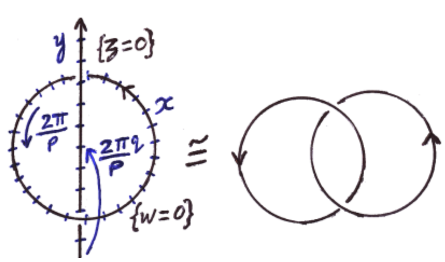

Figure 16: The great circles $\{z = 0\}$ and $\{w = 0\}$ in $S^3 \subset \mathbb{C}^2$ with this orientation are together homeomorphic to the right Hopf link.

Then define:

$$\begin{aligned} V &= \{(z,w) \in S^3 : |w| \le 1\}, \\ W &= \{(z,w) \in S^3 : |z| \le 1\}, \\ T &= V \cap W \\ &= \{(z,w) \in S^3 : |z| = |w| = 1\} \cong \mathbb{T}^2. \end{aligned}$$

Recall that $D \times S^1$ is a *solid torus.* We refer to any curve of the form $\partial D \times \{z\} \subset D \times S^1$ as a *meridian.* Now, as indicated in Figure 17 we take μ and λ to be generators of $\pi_1(T)$. Thus μ and λ are meridians of the solid tori V and W, respectively. We give μ and λ the orientations shown in Figure 17.

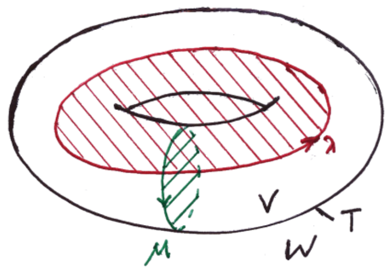

Figure 17: The curves μ and λ are oriented so that μ, λ and the outward normal for V form a right-handed frame.

Definition 15.7. Write $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{\alpha \in \mathbb{C} : \alpha^p = 1\}$ for $p \neq 0$, and fix $q \in \mathbb{Z}$ with $\gcd\{q, p\} = 1$. This acts on S^3 via:

$$\alpha \cdot (z, w) = (\alpha z, \alpha^p w).$$

Definition 15.8. Define $L(p, q) = \mathbb{Z}_p \backslash S^3$, the (p, q)–*lens space.*

Exercise 15.7. $L(p, q)$ is an orientable 3–manifold.

Example 15.8. We have $L(1, 0) = S^3$.

Exercise 15.8. Show that $L(2, 1) \cong P^3$.

Proposition 15.4. *Suppose $V, W \cong \mathbb{D}^2 \times S^1$ and $\varphi\colon \partial W \to \partial V$ is a homeomorphism. Show that $M = V \cup_\varphi W$ is either a lens space or is $S^1 \times S^2$.*

Note. We have $\pi_1(L(p, q)) \cong \mathbb{Z}_p$. Thus if $L(p', q') \cong L(p, q)$ then $p' = p$.

Exercise 15.9. Show that if $q' = \pm q^{\pm 1}$ modulo p, then $L(p, q') \cong L(p, q)$.

Remark. The converse holds, but is much harder to prove (see Brody 1960).

Remark. Whitehead (1941) showed that $L(p, q) \simeq L(p, q')$ (the spaces are homotopy equivalent) if and only if $qq' = \pm k^2$ modulo p for some k.

Example 15.9. We have $L(7, 1) \simeq L(7, 2)$, but these spaces are not homeomorphic.