MA4J2 Three Manifolds

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today

Lecture 16

For lens spaces, we have the following definitions:

- The quotient space $\mathbb{Z}_p \backslash S^3$.
- The gluing $V \cup_\varphi W$, the union of solid tori, which is either a lens space or $S^2 \times S^1$.
- The following construction: let $B = \{(z,t) \in \mathbb{C} \times \mathbb{R} : \; |z|^2 + t^2 \leq 1\}$, a 3–ball. Let $D^\pm$ be the upper (respectively lower) hemisphere of ∂B, as in Figure 1.

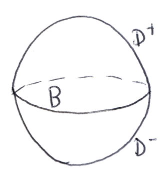

Figure 1: $D^\pm$ are the upper and lower hemispheres of ∂B.

Fix $\alpha = \exp(2\pi i/p)$ and glue D^- to D^+ by $\varphi\colon D^- \longrightarrow D^+$, where $\varphi(z,t) = (\alpha^q z, -t)$. See Figure 2.

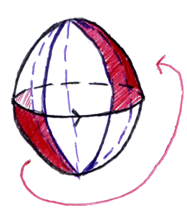

Figure 2: The lower hemisphere is glued to the upper by a $2\pi \cdot q/p$ twist.

Notice that, as Figure 2 indicates, there is a nice triangulation of B by a collection of p tetrahedra, all sharing the z–axis as an edge. Notice also that a neighborhood of the midpoint of any edge is a half-ball

$$B^3_+ \cong \{(x,y,z):\ z \geq 0,\ x^2+y^2+z^2 \leq 1\}$$

and p copies of these are glued, each to the next. So "geometrically", an edge has $p\pi$ dihedral angle which is $(p-2)\pi$ too much. So we consider a lens with dihedral angle $2\pi/p$ at the equator, as in Figure 3.

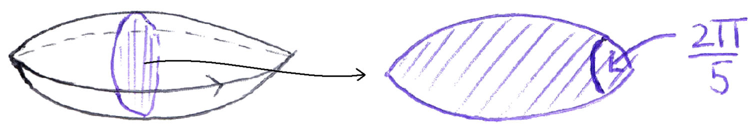

Figure 3: A lens with dihedral angle $2\pi/p$ at the equator (here, $p = 5$).

Now we can glue and get the right amount of dihedral angle. More precisely, the lens should live in S^3 and be cut out by great hemispheres, each meeting the next at angle $2\pi/p$. In Figure 4, you can see the lenses for $p = 10$. Glue pairs of these together to get lenses for $p = 5$.

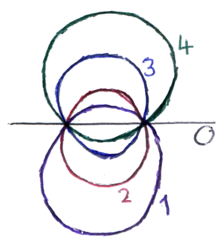

Figure 4: 10 copies of the lens tile S^3.

Exercise 16.1. Check that the three definitions agree.

Recall that we defined the meridian and longitude μ, λ for the torus $T = V \cap W \subset S^3$. See Figure 5.

Definition 16.1. If $K = s\mu + r\lambda$ then the *slope* of K is r/s.

Let $K = s\mu + r\lambda \in \pi_1(T)$, a simple closed curve. In Figure 6 for example, $K = 3\mu + 2\lambda$ has slope $2/3$ in T.

Notation. For $\alpha, \beta \in \pi_1(T)$ we define $\alpha \cdot \beta$ to be the signed intersection number. So

$$\begin{array}{cc} \mu \cdot \mu = 0 & \mu \cdot \lambda = +1 \\ \lambda \cdot \mu = -1 & \lambda \cdot \lambda = 0 \end{array}$$

and thus $\mu \cdot K = r$ and $K \cdot \lambda = s$.

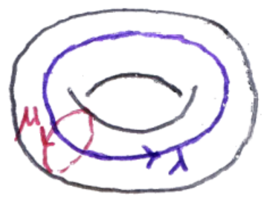

Figure 5: The torus T with meridian μ and longitude λ. Note that the orientation of μ, that of λ, and the outward normal to V, in that order, obey the right-hand rule.

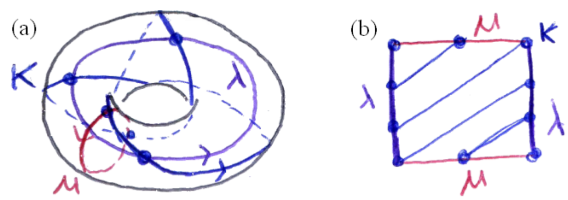

Figure 6: The right handed trefoil knot K has slope 2/3. (a) K as seen in the torus T, and (b) K as seen in $\mathbb{R}^2/\mathbb{Z}^2 \cong T$.

Definition 16.2. Suppose $r, s \in \mathbb{Z}$ are coprime, with $|r|, |s| > 1$. We call $K = r\lambda + s\mu \subset T \subset S^3$ the (r,s)*–torus knot*. Then we define $X_K := S^3 - n(K)$, the *knot exterior*. Moreover, we define $V_K := V - n(K)$, $W_K := W - n(K)$ and $A = T_K = T - n(K)$.

In Figure 7, z is the core curve of $A = V_K \cap W_K$.

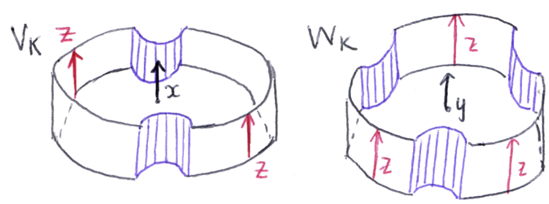

Figure 7: The cross-sections of V_K and W_K. The loop z is the core curve of $A = V_K \cap W_K$, and the loops x and y are the generators of $\pi_1(V_k)$ and $\pi_1(W_k)$ respectively.

Recall that the inclusions $i : A \hookrightarrow V_K$ and $j : A \hookrightarrow W_K$ induce maps i_* and j_* giving the following diagram:

$$\begin{array}{ccc} & \pi_1(A) = \langle z \rangle & \\ \swarrow i_* & & j_* \searrow \\ \pi_1(V_k) = \langle x \rangle & & \pi_1(W_k) = \langle y \rangle \end{array}$$

Exercise 16.2. Show that $i_*(z) = x^r$ and $j_*(z) = y^s$ hence i_* and j_* are injective, where x and y are the loops shown in Figure 7.

By Seifert-van Kampen, assuming that $r, s \neq 0$, we get the following push-out where the lower maps are again inclusions:

$$\begin{array}{ccccc} & & \pi_1(A) = \langle z \rangle & & \\ & \swarrow^{i_*} & & \searrow^{j_*} & \\ \pi_1(V_k) = \langle x \rangle & & & & \pi_1(W_k) = \langle y \rangle \\ & \searrow & & \swarrow & \\ & & \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} \cong \langle x, y \,|\, x^r = y^s \rangle =: \Gamma_{r,s} & & \end{array}$$

Via group theory, one can show that $\Gamma_{r,s} \cong \Gamma_{p,q}$ if and only if $\{|p|, |q|\} = \{|r|, |s|\}$.

Lecture 17

Aside. Note that

- $SO(2) \cong S^1$,
- $SO(3) \cong \mathbb{P}^3$ and
- $SL(2, \mathbb{R}) \cong \mathrm{int}(\mathbb{D} \times S^1) \cong \mathbb{R}^2 \times S^1$, the latter is not an isomorphism of groups.

Remark. We now have the following remarkable fact. Let $K \subset S^3$ be the trefoil knot and define $Y_K = S^3 - K$ be the *knot complement*, an open three-manifold. Then Y_K is homeomorphic to $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$.

For the following we assume that K is not the unknot, i.e. $|p|, |q| \geq 2$.

Theorem 17.1. *Suppose $K = K_{p,q}$ is the (p, q)–torus knot, then the annulus $A = T - n(K)$ is the unique* essential *annulus in X_K, up to isotopy.*

We will prove this later in the course.

Corollary 17.1. *Define $X_{p,q} = X_K$, where $K = K_{p,q}$. Then $X_{p,q} \cong X_{r,s}$ if and only if $\{|p|, |q|\} = \{|r|, |s|\}$.*

Non-uniqueness of torus decompositions

Now we closely follow Hatcher. Let $V_i \cong \mathbb{D} \times S^1$, $i = 1, 2, 3, 4$. Let $A_i \subset \partial V_i$ be an embedded annulus and suppose A_i winds q_i times about V_i with $q_i \geq 2$; for examples see Figure 8.

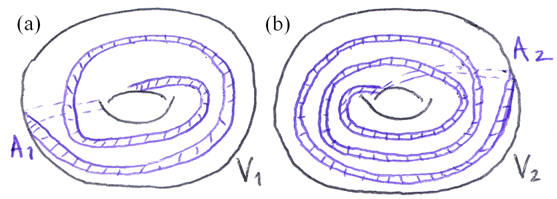

Figure 8: Two examples of a winding annulus; in (a) $q_1 = 2$ and in (b) $q_2 = 3$.

Another way to define q_i is the following: Let α_i be a core curve of A_i and define q_i via $q_i = |\alpha_i \cdot \partial D_i|$. Let $A_i' = \overline{\partial V_i - A_i}$ and pick $\varphi\colon A_i' \longrightarrow A_{i+1}$ where we take the indices modulo 4. Let $M = \sqcup V_i/\varphi_i$; see Figure 9(a). Let B_i denote the image of A_i in M. Now we define $M_i = V_i \cup_{\varphi_i} V_{i+1}$. Let $T_1 = B_1 \cup B_3$ and $T_2 = B_2 \cup B_4$. Thus $M = M_1 \cup_{T_1} M_3 = M_2 \cup_{T_2} M_4$.

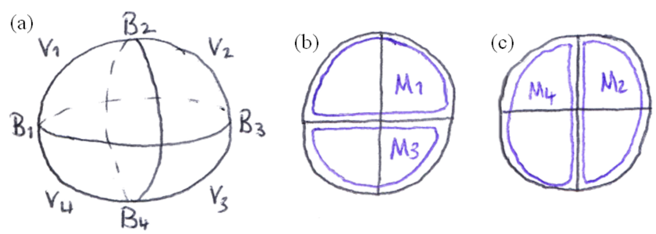

Figure 9: (a) A schematic of M. B_i is the image of A_i in M. (b) and (c) are schematics of two different torus decompositions.

Finally, we claim that $B_1 \cup B_3$ and $B_2 \cup B_4$ are incompressible tori in M. If we now choose the q_i to all be distinct and coprime then, for $i = 1, 2, 3, 4$, then manifold M_i is a torus knot exterior. So we have, for these choices of q_i, that M_1 is not homeomorphic to M_2 or M_4 and M_3 is not homeomorphic to M_2 or M_4. Thus the torus decompositions T_1 and T_2 are different; see Figures 9(b) and (c).

Remark (17.2)**.** This requires the following facts. If $X_{p,q} = S^3 - n(K_{p,q})$, then

- $\partial X_{p,q}$ is incompressible,
- $X_{p,q}$ is atoroidal and
- Theorem 17.1.

We will prove these facts later. To do so, and so to understand the non-uniqueness of torus decompositions, we must first understand *Seifert fibred spaces*.

Fibred solid tori

Fibre $D \times I$ by intervals of the form $\{x\} \times I$. We call $\{0\} \times I$ the *central fibre*. Let $\varphi\colon D \times \{1\} \longrightarrow D \times \{0\}$ be a $2\pi q/p$ rotation, $\varphi(z,1) = (\alpha^q z, 0)$ where as usual p and q are coprime. Define $V_{p,q} = D \times I/\varphi$, the (p,q)–*fibred solid torus.* Notice that $\{0\} \times I$ now gives a circle as does the set of fibres $\{\alpha^k \cdot (z \times I) : \ \alpha^p = 1\}$. Note that $V_{p,q}$ is given a *fibring*, i.e. a decomposition into circles.

Definition 17.3. A *Seifert fibring* of a three-manifold M is a partition $\mathcal{F}$ of M into circles (the *fibres*) such that every fibre $\lambda \in \mathcal{F}$ has arbitrary small regular neighbourhoods $N(\lambda)$ all homeomorphic to $V_{p,q}$ for some fixed p, q. Here the homeomorphisms are all fibre-preserving.

Remark. The integers p, q only depend on λ.

Definition 17.4. We call p the *multiplicity* of λ.

Note that the space $V_{p,q}$ is Seifert fibred itself and the central fibre has multiplicity p while all other fibres have multiplicity equal to 1.

Definition 17.5. If λ has multiplicity greater than 1, then we call λ a *singular* fibre. All other fibres are called *generic*. See Figure 10.

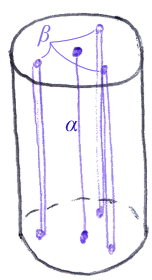

Figure 10: Inside of $V_{3,1}$ the central fibre α is singular (with multiplicity three) while all others, for example β, are generic.

Lecture 18

Exercise 18.3. If M is compact then there are only finitely many singular fibres, all contained in the interior of M.

Exercise 18.4. Show that $L_{p,q}$ is a Seifert fibred space with at most two singular fibres. Compute their multiplicities.

Exercise 18.5. Let $K = K_{p,q}$ be the (p,q)–torus knot. Show that X_K is a Seifert fibered space. Find the singular fibres and their multiplicities.

Example 18.1. Let $M = V_1 \cup V_2 \cup V_3 \cup V_4$ as in the last lecture. Then M is a Seifert fibred space with 4 singular fibres.

Definition 18.6. Suppose $(M, \mathcal{F})$ is a Seifert fibred space. Let $B = M/S^1$ be the *base orbifold*; that is, the quotient of M sending fibres to points.

Example 18.2. Suppose $M = V_{p,q}$. The quotient M/S^1 is a disk D with a cone point at the centre. The angle at the cone point is $2\pi/p$; see Figure 11.

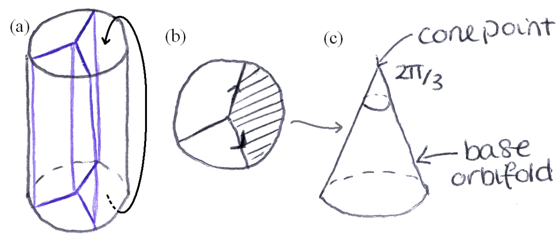

Figure 11: (a) The solid torus $V = V_{3,1}$. (b) A meridian disk for V. (c) The quotient V/S^1 is a cone with angle $2\pi/3$ at the cone point.

Exercise 18.6. In exercises 18.1 and 18.2, identify the base orbifolds.

Example 18.3. Notice that if $\rho\colon T \longrightarrow F$ is an S^1–bundle then $T/S^1 \cong F$.

Theorem 18.2 (1.9 in Hatcher)**.** *Let M be compact, irreducible and orientable. There exists a torus system $T \subset M$ such that all components of $M - n(T)$ are either atoroidal or Seifert fibred spaces. Furthermore any minimal such system is unique up to isotopy.*

Remark. The example from last lecture, M, contains infinitely many non-isotopic incompressible tori. Hence the uniqueness of Theorem 18.2 requires that we not cut along tori in Seifert fibred spaces.