MA4J2 Three Manifolds

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Lecture 19

Suppose $F \subset M$ is properly embedded, and M is compact, irreducible and orientable. Recall that $(D, \partial D) \subset (M, F)$ is a *surgery disk* for F if $D \cap F = \partial D$. D is *trivial* if ∂D bounds a disk in F. If D is not trivial, then it is a *compressing disk* for F.

Definition 19.1. A disk D with $\partial D = \alpha \cup \beta$ such that α and β are connected and $\alpha \cap \beta = \partial\alpha = \partial\beta$ is a *bigon*; see Figure 1.

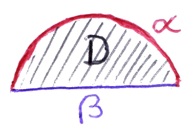

Figure 1: A bigon D.

Definition 19.2. Say $D \subset M$ is a *surgery bigon* for $F \subset M$ if D is a bigon, $D \cap F = \alpha$ and $D \cap \partial M = \beta$. Say that D is *trivial* if there is a bigon $D' \subset F$ so that $\partial D' = \alpha' \cup \beta'$, $\alpha = \alpha'$ and $D' \cap \partial M = \beta'$, as in Figure 2. If D is not trivial, call it a *boundary compressing bigon*, or simply a *boundary compression*.

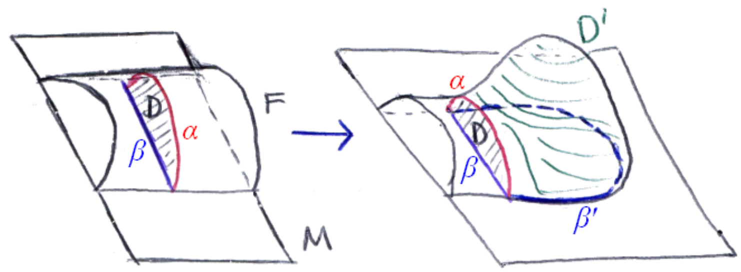

Figure 2: D is a trivial surgery bigon. Note that D' is not properly embedded in M but contained entirely in F.

Recall that a two-sided simple closed curve $\alpha \subset F^2$ is *essential* if α does not bound a disk on either side (Figure 3(a)). A sphere $S \subset M^3$ is *essential* if it does not bound a three-ball on either side (Figure 3(b)). If M is irreducible then a disk $(D, \partial D) \subset (M, \partial M)$ is *essential* if ∂D is essential in ∂M (Figure 3(c)).

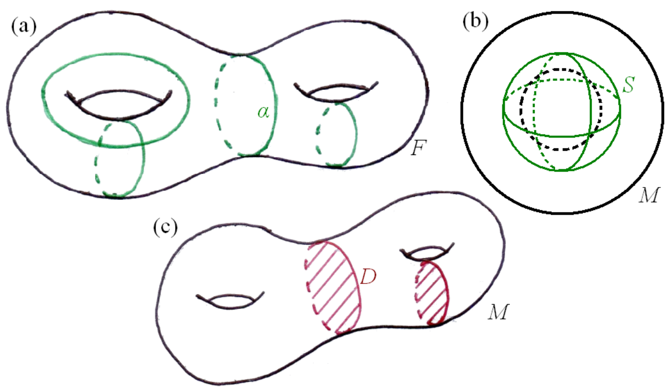

Figure 3: (a) All the green curves here are essential. (b) Here S is essential in M. (c) These disks are essential in M.

Definition 19.3. Suppose that $S \subset M$ is a properly embedded, connected, two-sided surface that is not a disk or a sphere. We say S is *essential* if it is incompressible and boundary incompressible.

Definition 19.4. If all surgery disks are trivial, we call F *incompressible*; similarly, if all surgery bigons are trivial, call F *boundary incompressible*.

Exercise 19.1. Suppose $S \subset M$ is an essential surface. Show that $\partial S \subset \partial M$ is essential.

Proposition 19.1. *If $S \subset D^2 \times S^1$ is essential then S is isotopic to $D^2 \times \{z\}$ for some $z \in S^1$.*

Proof. Let $\mu_z = \partial D^2 \times \{z\}$. We call μ_z the meridian curves. Abusing notation, let $D = D^2 \times \{1\}$. Then by Exercise 19.1, ∂S is essential so we may isotope components of ∂S so that all are either equal to meridian curves, or are transverse to all meridian curves, as in Figures 4(a) and (b).

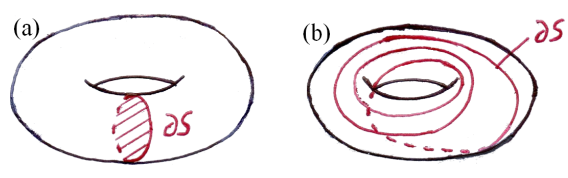

Figure 4: (a) Here the component of ∂S is meridian curve. (b) Here ∂S is transverse to all meridian curves.

Thus, we may assume that ∂S is transverse to μ_1, and via isotopy relative to ∂M, we may assume that S is transverse to D. Then $S \cap D$ is a collection of arcs and loops, as in Figure 5.

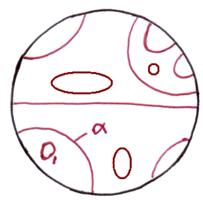

Figure 5: $S \cap D$ is a collection of arcs and loops.

We proceed as follows:

Step 1: First suppose $\alpha \subset D \cap S$ is an innermost loop, so α bounds a disk $D_1 \subset D$ such that $D_1 \cap S = \partial D_1$. So D_1 is a surgery disk for S and thus, as S is incompressible, there is a disk $E \subset S$ with $\partial E = \partial D_1 = \alpha$, as in Figure 6(a). So $D_1 \cup E$ is a 2–sphere. As $D \times S^1$ is irreducible, $D_1 \cup E$ bounds a 3–ball B, so there is an isotopy supported in $n(B)$ moving E past D_1; see Figure 6(b). This gives an isotopy of S, reducing $|S \cap D|$. So without loss of generality, we may assume that $D \cap S$ consists only of arcs.

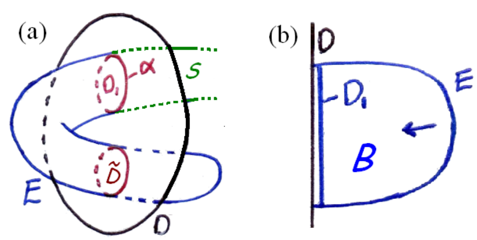

Figure 6: (a) $E \cup D_1$ bounds a 3–ball B, so (b) we may isotope E through $n(B)$ past D_1 to reduce $|S \cap D|$.

Step 2: Now suppose $\alpha \subset D \cap S$ is an outermost arc. So α cuts off from D a surgery bigon D_1. Since S is boundary incompressible, α cuts off a bigon E from S. Let $\gamma = E \cap \partial(D \times S^1)$ and $\beta = D_1 \cap \partial(D \times S^1)$. See Figure 7.

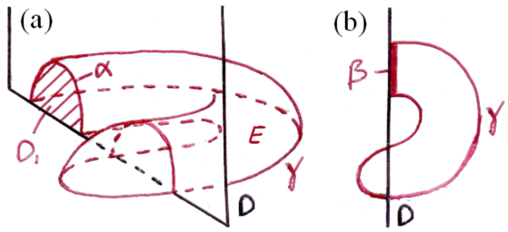

Figure 7: (a) α cuts a surgery bigon D_1 from D and E from S. (b) A plan view of (a).

Notice that $D_1 \cup E$ is a disk, with $D_1 \cap E = \alpha$. Thus $D_1 \cup E$ lifts to $\widetilde{D \times S^1} \cong D \times \mathbb{R}$, as in Figure 8.

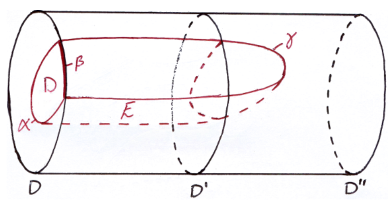

Figure 8: $D_1 \cup E$ lifts to $\widetilde{D \times S^1} \cong D \times \mathbb{R}$.

Let $h : D \times \mathbb{R} \to \mathbb{R}$ be projection to the second factor, and notice that:

$$h(\partial_+\gamma) = h(\partial_-\gamma)$$

as $\partial_\pm\gamma \in \partial D$. So by Rolle's theorem, $(h|\gamma)'$ has a zero, so γ is not transverse to μ_z for some $z \in S^1$, giving a contradiction. Thus without loss of generality, we may assume $S \cap D = \varnothing$.

Step 3: Next, define $B = (D \times S^1) - n(D)$. This is a 3–ball, and $S \subset B$. Pick any component $\delta \subset \partial S$. So δ divides ∂B into disks C and C'. So push C, say, into B, keeping ∂C inside of S. This gives a disk in the interior of B. See Figure 9. If $C \cap S \neq \partial C$, then we may isotope S, as in Step 1, to reduce $|S \cap C|$. So C gives a surgery disk for S. Thus S is a disk.

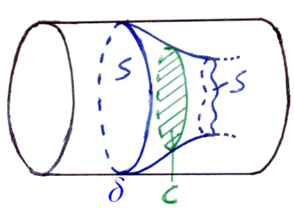

Figure 9: Push C into B (keeping ∂C inside of S) to get a disk in the interior of B.

Finally, Alexander's theorem implies that S is isotopic to $D \times \{z\}$ for some $z \in S^1$, fixing δ pointwise. □

Note. All surgery disks for S^2 are trivial, and all surgery disks and bigons for $\mathbb{D}^2$ are trivial, hence they are excluded from the statement of Proposition 19.1.

Definition 19.5. Suppose $(\alpha, \partial\alpha) \subset (A^2, \partial A^2)$ is an arc in an annulus. It is *trivial* if it cuts a bigon off of A, and *essential* otherwise. See Figure 10.

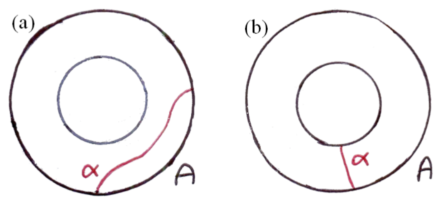

Figure 10: (a) A trivial arc. (b) An essential arc.

Lecture 20

Exercise 20.2. Suppose $F \subset M$ is two-sided and incompressible. Suppose $D \subset M$ is a surgery bigon for F and suppose F_D is the result of surgery. Show that $F_D \subset M$ is incompressible.

Exercise 20.3. Deduce from the above that if $\rho : T \to F$ is an I–bundle then $\partial_h T$ is boundary incompressible.

Lemma 20.2 (1.10 in Hatcher)**.** *Suppose that $S \subset M$ is a connected, two-sided, incompressible surface, and M is irreducible. Suppose S admits a boundary compressing bigon D with $\partial D = \alpha \cup \beta$, $\alpha = D \cap S$, $\beta = D \cap \partial M$ and β is contained in a torus component $T \subset \partial M$. Then S is a boundary parallel annulus.*

Proof. By Exercise 19.1, $\partial S \cap T$ is essential in T. Let $A = T - n(\partial S)$, so A is a collection of annuli. So $\beta \subset A$ is either trivial or essential, as in Figure 11(a).

Case 1: Suppose that $\beta \subset A$ is trivial. So β cuts a bigon E off of A. Then $D \cup E$ is a disk. Isotope $D \cup E$, keeping $\partial(D \cup E)$ in S, to get a surgery disk for S; see Figure 11(b).

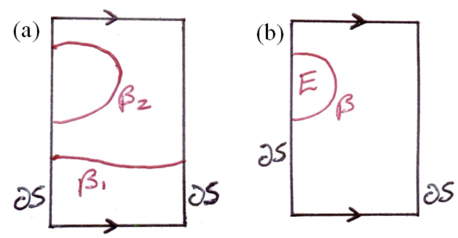

Figure 11: (a) β_1 is essential while β_2 is trivial. (b) Trivial arcs define a surgery bigon for S.

Since S is incompressible, $D \cup E$ cuts a disk D' out of S, and hence D was a trivial surgery bigon, as in Figure 12.

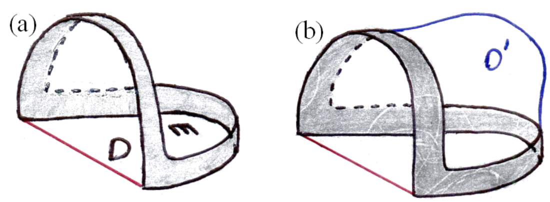

Figure 12: $D \cup E$ cuts a disk D' from S and so D is trivial.

Case 2: Suppose β is essential in A. If ∂B is contained in a single component of ∂S, then S is one-sided, giving a contradiction. To see this, we can orient β and ∂S so that both intersections have positive sign, as in Figure 13.

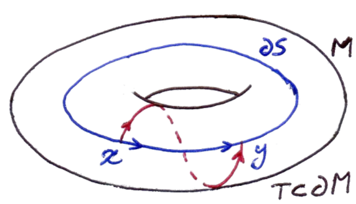

Figure 13: We can orient β and ∂S so that both intersections have positive sign.

Then following α we find that S is one-sided, as in Figure 14.

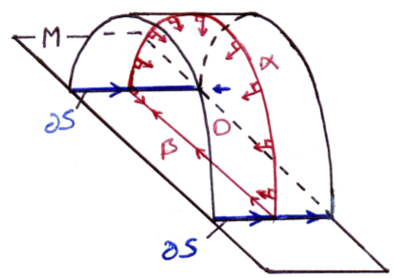

Figure 14: Carrying the orientation along α gives a different orientation to carrying along ∂S, a contradiction.

So we have that β connects distinct components of ∂S, as in Figure 15.

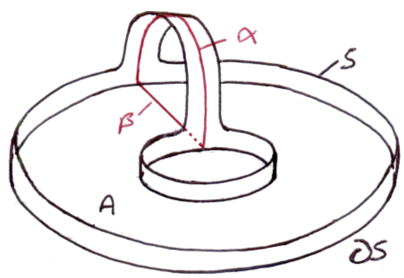

Figure 15: β connects distinct components of ∂S.

Boundary compress S along D to get S_D. Note that S_D is incompressible, by Exercise 20.2, and that S_D has a trivial boundary component, so S_D is a disk. To see this, say ∂S_D bounds E in T. So isotope E into E' in M, keeping ∂E in S_D, as in Figure 16.

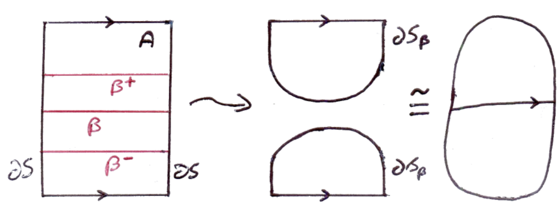

Figure 16: Cutting along β gives two components of ∂S_β, and the identification gives a trivial curve in ∂S_D.

Since S_D is incompressible, $\partial E'$ must cut a disk out of S_D, so S_D is a disk. Since M is irreducible, S_D is boundary parallel; in fact it is parallel to the original E.

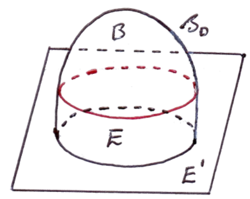

Figure 17: S_D is boundary parallel.

So S_D cuts a 3–ball B out of M. Letting $V = B \cup N(D)$, this is a solid torus, giving a parallelism of S with the annulus A, as in Figure 18. $\square$

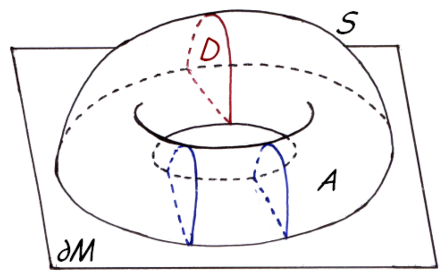

Figure 18: S is boundary parallel to the annulus A.

Definition 20.6. Suppose that $(M, \mathcal{F})$ is Seifert fibred. Then we say that a properly embedded surface $S \subset M$ is *vertical* if S is a union of fibres, and it is *horizontal* if S is transverse to the fibres. We make the same definitions for $S \subset T$ for an I–bundle $\rho : T \to F$.

Exercise 20.4. All essential surfaces $S \subset T$, where $\rho : T \to F$ is an I–bundle, are isotopic to either vertical or horizontal surfaces.

Lecture 21

Lemma 21.3 (1.11 in Hatcher)**.** *Suppose that $(M, \mathcal{F})$ is compact, connected and irreducible. Supppose $S \subset M$ is essential. Then after a proper isotopy, S is either vertical or horizontal.*

Proof. Let $Z := \{\alpha_i\}_{i=1}^k$ be the set of singular fibres of $\mathcal{F}$; if M has no singular fibres, and $\partial M = \varnothing$, then let $\{\alpha_1\}$ be a single generic fibre. Let $M_0 = M - n(Z)$. Let $B = M/S^1$ and let $B_0 = M_0/S^1$. Note that $\partial B_0 \neq \varnothing$. In fact B_0 is B with neighbourhoods of cone points removed, as in Figure 19.

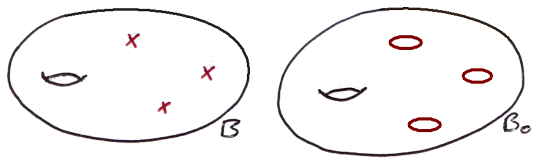

Figure 19: B_0 is B with neighbourhoods of cone points removed.

Example 21.1. If $M = V_{p,q}$ then Z is just the central fibre. Then $M_0 = \mathbb{A}^2 \times S^1$ and $B_0 = \mathbb{A}^2$; See Figure 20.

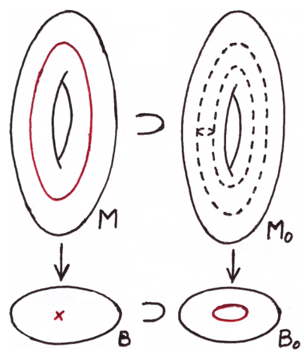

Figure 20: $M_0 = \mathbb{A}^2 \times S^1$ and $B_0 = \mathbb{A}^2$.

Choose a system of arcs in B_0 cutting B_0 into a disk, i.e. as in Figure 21.

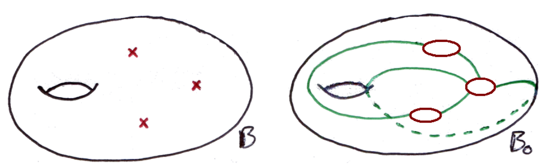

Figure 21: We may choose a system of arcs cutting B_0 into a disk.

Let $A \subset M_0$ be the vertical annuli above this system of arcs. So $M_0 - n(A) =: M_1$ is a solid torus, fibred by $\mathcal{F}|M_1$, with all fibres generic. Given an essential surface S, all components of ∂S are essential in ∂M.

(i) We may isotope them to all be vertical or horizontal with respect to the fibring $\mathcal{F}|\partial M$.

(ii) Isotope S (relative to ∂S) so that S meets Z transversely, and so meets $n(Z)$ in horizontal disks. Define $S_0 = S \cap M_0$, and make S_0 intersect A transversely. Consider the arcs and loops of $S_0 \cap A$, as in Figure 22.

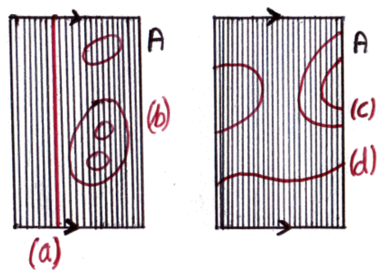

Figure 22: (a) An essential loop. (b) Trivial loops. (c) Trivial arcs. (d) An essential arc.

(iii) If there is a trivial loop, then there is an innermost such. Now, using incompressibility of S and irreducibility of M, there is an isotopy of S reducing $|S \cap A|$ as usual. So without loss of generality, there are no trivial loops.

(iv) Suppose $\beta \subset S \cap A$ is an outermost trivial arc and let D be the bigon cut our of A by β. If $\partial\beta \subset \partial M$ then D is a surgery bigon for S, but as in Proposition 19.1, ∂S is either contained in or transverse to $\mathcal{F}|\partial M$, giving a contradiction. To see this, since S is boundary incompressible, there is a bigon E contained in S, as in Figure 23.

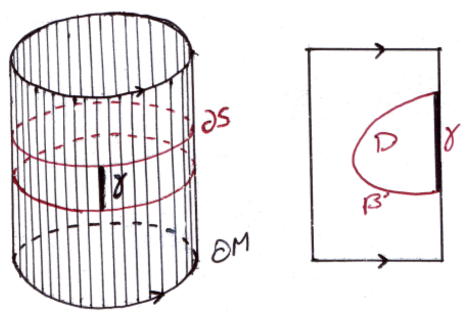

Figure 23: γ is parallel to the fibres.

So letting $\partial E = \beta \cup \gamma'$, we find that γ' is not transverse to $\mathcal{F}|\partial M$. On the other hand, if $\partial\beta \subset \partial M_0 - \partial M$, then a baseball move across D reduces $|S \cap (Z)|$ by two. Now without loss of generality, every component of $S \cap A$ is either horizontal or vertical.

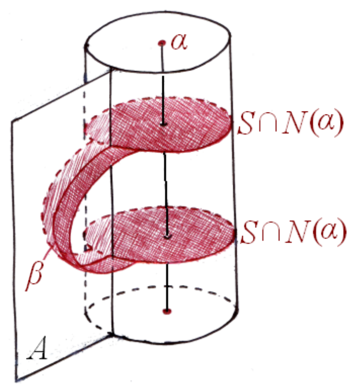

Figure 24: A baseball move across α reduces $|S \cap Z|$ by 2.

(v) Define $S_1 = S_0 \cap M = S_0 - n(A)$. So $\partial S_1 \subset M_1$ is completely horizontal or completely vertical. We may assume that S_1 is incompressible in M_1. Thus S_1 is either a collection of horizontal meridian disks, or a collection of boundary parallel annuli. If S_1 contains an annulus with slope that of the meridian, then S_1 is compressible. If S_1 contains an annulus $B \subset S_1$ with ∂B horizontal, then we see a surgery bigon with vertical boundary. So do a baseball move and return to case (iv).

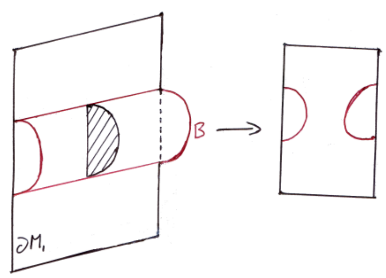

Figure 25: If S_1 contains an annulus B with ∂B horizontal, we may do a baseball move and reduce to case (iv).

So S_1 is now a collection of horizontal meridian disks, or a collection of boundary parallel vertical annuli. It follows that S_0, and so S, is either horizontal or vertical. □

Remark. Vertical surfaces are easy to classify. They are orientable or not, and the base is I or S^1.

Base Orbifold	I	S^1	
	A^2	T^2	orientable
	M^2	K^2	non-orientable