MA4J2 Three Manifolds

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Lecture 22

Notation. Suppose F is not orientable. Let $F \tilde{\times} I$ denote the orientation I–bundle over F. Likewise define $F \tilde{\times} S^1$.

Exercise 22.1. Show that $P^2 \tilde{\times} I$ is homeomorphic to $P^3 - \text{int}(B^3)$.

We now discuss orbifolds.

Definition 22.1. We say that $B = (S, Z)$ is an 2–*orbifold* if S is a surface and $Z \subset \text{int}(S)$ is a finite set such that for every $z \in Z$ we have an *order* $p_z \in \mathbb{Z}_+$. We call Z the *singular set*. A point $z \in Z$ is a *cone point* if $p_z > 1$.

Example 22.1. A surface is an orbifold with $Z = \varnothing$.

Example 22.2. The square pillow case, $S^2(2, 2, 2, 2)$, shown in Figure 1, is an orbifold.

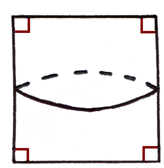

Figure 1: A picture of the square pillow case $S^2(2, 2, 2, 2)$.

Definition 22.2. If S is a surface with a triangulation T then we define the *Euler characteristic* of S to be $\chi(S) = V - E + F$ where V denotes the number of vertices, E the number of edges and F the number of triangles (faces).

Exercise 22.2. Show that χ stays unchanged under the Pachner moves. Figure 2 shows the Pachner moves. Since any two triangulations of a fixed closed surface are related by Pachner moves, the Euler characteristic is independent of the choice of T.

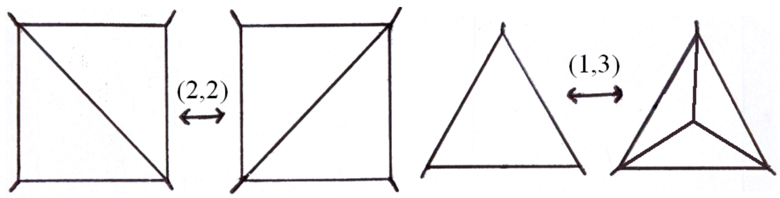

Figure 2: The Pacher moves.

Example 22.3. You can see by the triangulation shown in Figure 3(a) that $\chi(S^2) = 4 - 6 + 4 = 2$. Similarly, Figure 3(b) shows that $\chi(T^2) = 1 - 3 + 2 = 0$.

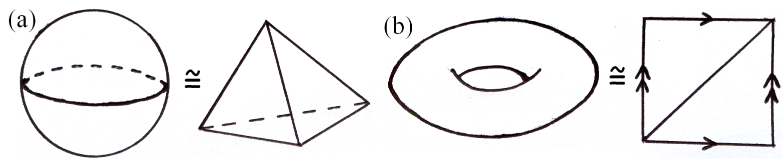

Figure 3: (a) A triangulation of a 2–sphere. (b) A triangulation of the 2–dimensional torus.

Definition 22.3. We define the *Euler characteristic* of an orbifold via

$$\chi_{\text{orb}}(B) = \chi(S) + \sum_{z \in Z} \left(\frac{1}{p_z} - 1 \right).$$

Example 22.4. $\chi_{\text{orb}}(S^2(2,2,2,2)) = 2 + 4(1/2 - 1) = 0$.

Exercise 22.3. List all 2–orbifolds B so that $\chi_{\text{orb}}(B) = 0$.

Exercise 22.4. What can you say about B so that $\chi_{\text{orb}}(B) > 0$?

Orbifold Covers

Example 22.5. The map from $D \subset \mathbb{C} \to D$ which sends z to z^n is an orbifold map of order n. In Figure 4, $n = 3$.

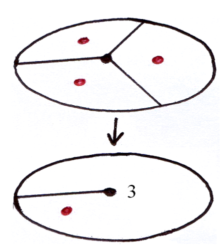

Figure 4: The map $z \mapsto z^3$ from $D \subset \mathbb{C}$ to itself is a three-fold cover.

Definition 22.4. If C, B are 2–orbifolds then $\varphi\colon C \to B$ is a *cover* if

1. $\varphi^{-1}(Z_B) = Z_C$,
2. $\varphi|(C - Z_C)\colon C - Z_C \to B - Z_B$ is a d–fold cover and
3. for every point $z \in Z_B$, we have $d/p_z = \sum_{y \in \varphi^{-1}(z)} 1/p_y$.

Note that φ restricted to any regular neighbourhood of a point $z \in Z_C$ is modelled on the example $z \longmapsto z^n$.

Example 22.6. The quotient of T^2 via the 180° rotation shown in Figure 5 is a degree two orbifold cover.

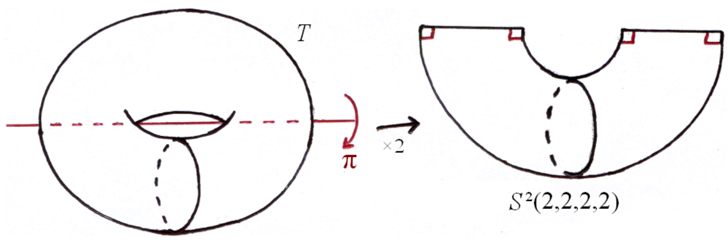

Figure 5: The quotient map of the 2–dimensional torus via the 180° rotation.

Exercise 22.5. Show that if $\varphi\colon C \to B$ is a d–fold orbifold cover then $\chi_{\rm orb}(C) = d \cdot \chi_{\rm orb}(B)$. As warm-up, show that if $\varphi\colon T \to S$ is a d–fold cover of surfaces then $\chi(T) = d \cdot \chi(S)$.

Exercise 22.6. List all 2–fold covers of $S^2(2, 2, 2, 2)$.

The following question is known as the *Hurwitz problem* and still open in general: Given B, C such that $\chi_{\rm orb}(C)/\chi_{\rm orb}(B) \in \{2, 3, 4, \ldots\}$ does there exists a d–fold cover?

Example 22.7. For $n \geq 2$, $S^2(n)$ is a *bad orbifold*, meaning it is not covered by a surface. Hence $S^2(n)$ is not covered by S^2. You can also see this because $2/(2 + (1/n - 1)) \notin \mathbb{N}$.

We now return to our original topic, horizontal surfaces. Suppose that $S \subset (M, \mathcal{F})$ is horizontal. As in the proof of Lemma 21.4, we may form $M \supset M_0 \supset M_1$ and $S \supset S_0 \supset S_1$. Let λ be any generic fibre and $d = |S \cap \lambda|$, so S_1 is a collection of d horizontal disks. Recall that Z is the set of all singular fibres. Thus $S \cap (N(Z))$ is also a collection of disks. Then S is formed by gluing horizontal disks along horizontal loops in $\partial N(Z)$ and horizontal arcs in A. Thus the quotient $\rho\colon M \to M/S^1 = B$ restricts to S to give a d–fold cover $\rho\colon S \to B$. So

$$\chi(S) = d \cdot \chi_{\rm orb}(B) = d \cdot \left(\chi(B) + \sum_{z \in Z} \left(\frac{1}{p_z} - 1 \right) \right).$$

Proof. See Hatcher. $\square$

Lecture 23

To answer the question of a student, we will expand the definition of a *boundary compression*.

Definition 23.5. Suppose $S \subset \partial M$ is a subsurface. Then we say S is *boundary compressible* if there is a bigon D with $\partial D = \alpha \cup \beta$ so that $D \cap S = \alpha$, $D \cap \overline{\partial M - S} = \beta$ and α does not cut a bigon out of S. Say that S is *boundary incompressible* if no such bigon exists.

Now we continue our discussion of horizontal surfaces. Suppose that $S \subset (M, \mathcal{F})$ is two-sided, horizontal and connected. Then we get the following corollary of Proposition 21.3 (1.11 in Hatcher).

Corollary 23.1. *The manifold $M - n(S)$ is an I–bundle.*

Proof sketch. Recall that S_1 was a collection of horizontal disks in $M_1 \cong D \times S^1$. So $n(S_1)$ cuts M_1 into cylinders foliated by intervals. The vertical sides of these solid cylinders glue to give the desired I–bundle. $\square$

Let $\rho \colon M - n(S) \longrightarrow F$ be the I–bundle map. Then there are two cases.

1. The manifold $M - n(S)$ is connected. So $M - n(S) \cong S \times I$ and thus $\partial_h(M - n(S)) = S \sqcup S$ and so $F \cong S$ and we find that M is an S–bundle over S^1. See Figure 6.

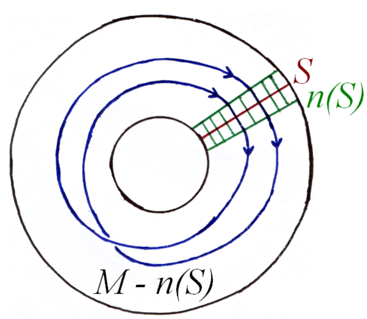

Figure 6: A picture of $M - n(S)$ as an S–bundle over S^1. The blue curve represents a generic fibre.

So the I–fibres in $N(S)$ and in $M - n(S)$ glue to give the Seifert fibring, $\mathcal{F}$. I.e., there is a *monodromy* (a homeomorphism $\varphi \colon S \longrightarrow S$) such that $M \cong S \times I/(x,1) \sim (\varphi(x), 0) =: M_\varphi$ and finally $S/\varphi \cong B$. The monodromy is *periodic* of period $d = |S \cap \lambda|$, i.e. $\varphi^d = \mathrm{Id}_S$. See Figure 7.

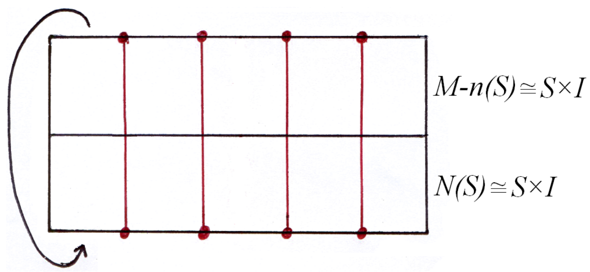

Figure 7: $M = (M - n(S)) \cup N(S) \cong M_\varphi$. Here φ has periodicity 4.

Example 23.8. Let φ be the hyperelliptic involution on the 2–torus shown in Figure 5. This is periodic.

Example 23.9. Glue the cube as shown in Figure 8 and note that planes parallel to the xy–plane glue to give tori.

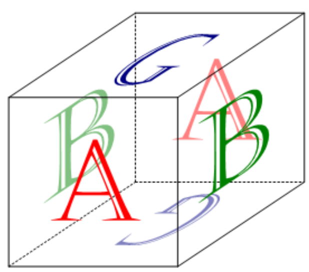

Figure 8: A cube with face pairings. The front and back are glued by the identity as are the left and right face. The bottom and top face are glued together by a 180° rotation.

Note that intervals parallel to the z–axis glue to give circles, 4 of length 1 and the rest of length 2.

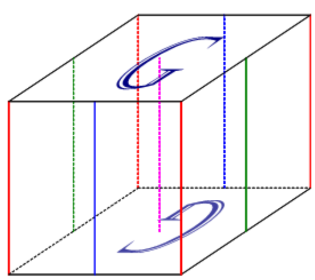

Figure 9: A picture of the different circles achieved by gluing intervals parallel to the z–axis. The gluings of the vertical faces are the same as in Figure 8 and are omitted.

All of the singular fibres in Figure 9 have length one, while all other vertical circles have length two. All other vertical circles have length 2. So $B \cong S^2(2,2,2,2)$ is the base orbifold, double covered by double covered by any horizontal surface, all of which are tori. See Figure 10.

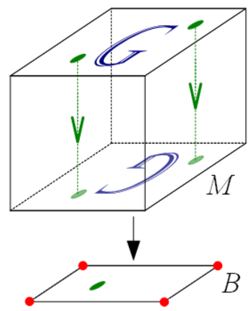

Figure 10: The base orbifold is a copy of the square pillow case: $B \cong M/S^1 \cong S^2(2,2,2,2)$, and double covered by T.

2. If $M - n(S)$ has two components then each is a twisted I–bundle over F and these glue to $N(S) \cong S \times I$ giving a *semibundle* (also called a *fibroid*). See Figure 11.

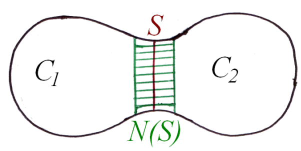

Figure 11: A picture of the two twisted I–bundles over F.

So letting T_1 and T_2 be the two I–bundles, we obtain M by gluing T_1 and T_2 to $N(S)$ and find involutions $\tau_i \colon S \longrightarrow S$ such that $T_i = S \times I/(x,0) \sim (\tau_i(x),0)$. Here the homeomorphism $\varphi = \tau_1 \circ \tau_2$ is again *periodic*.

Example 23.10. As an exercise, we showed that $P^3 - \text{int}(B^3) = P^2 \tilde{\times} I$. Here $\partial_h T \cong S^2$ and the involution τ is the antipodal map. So if we consider $T_1 \cup_S T_2$ where $T_i \cong P^2 \tilde{\times} I$, we find that $P^3 \# P^3$ is Seifert fibred. Check that $\tau_1 \circ \tau_2 = \tau^2 = \varphi = \text{Id}_S$ and so it is periodic.

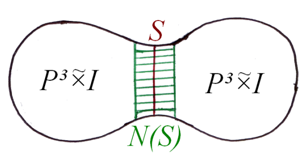

Figure 12: A picture of the gluing of $T_1 \cup_S T_2$.

Example 23.11. Consider the cube with face pairings given in Figure 3. Notice that the intervals parallel to the x–axis also define a Seifert fibring with $B = K^2$, the Klein bottle, and all fibres are generic, as in Figure 13(a). The planes $y = 1/4$ and $y = 3/4$ define a 2–torus $S \subset M$ and $M - n(S)$ has two components, both homeomorphic to $K \tilde{\times} I$.

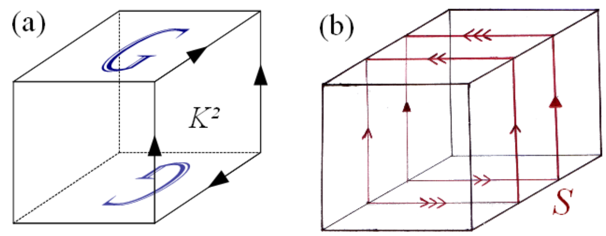

Figure 13: (a) Intervals parallel to the x–axis give a fibring with $B = K^2$. (b) Both components of $M - n(S)$ are homeomorphic to $K^2 \tilde{\times} I$.

Exercise 23.7. Check that these planes give a 2–torus with the claimed properties. Find the involutions τ_1, τ_2.

Lecture 24

Recall that every essential arc in $A^2 \cong S^1 \times I$ is isotopic to $\{\text{pt}\} \times I$, as in Figure 14.

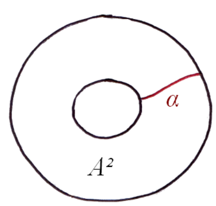

Figure 14: An essential arc in an annulus.

Exercise 24.8. Classify up to isotopy the essential arcs and loops in $\#_3 D^2$, the pair of pants.

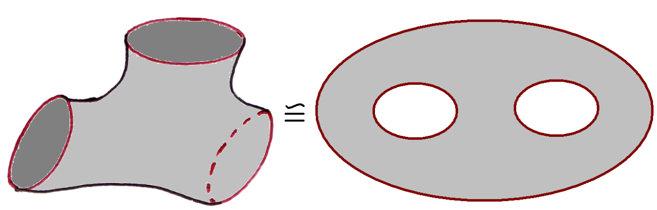

Figure 15: Two diagrams of the Pair of Pants.

Recall that if $X = X_K$ where $K = K_{p,q}$ is the (p,q)–torus knot then $B = X/S^1$ is the orbifold $D^2(p,q)$.

Exercise 24.9. Classify essential arcs and loops in $D(p,q)$. Deduce that the only essential vertical annulus in X is $A = V_K \cap W_K$. (Care is required if p or q is equal to 2, as then X contains a vertical Mobius band.)

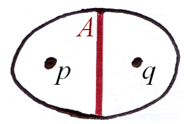

Figure 16: A diagram of $D^2(p,q)$. Note that the A here is the projection of the annulus into the orbifold.

Exercise 24.10. Use orbifold Euler characteristic to show that any horiontal surface $S \subset X$ has $\chi(X) \leq p + q - pq < 0$ as $p, q \geq 2$ and $p \neq q$. Deduce that X is atoroidal and A is the unique essential annulus in X, up to isotopy.

Exercise 24.11 (Harder)**.** Use Exercise 24.10 to prove that

$$g(K_{p,q}) = \frac{(p-1)(q-1)}{2}$$

where $g(K)$ is the minimal genus of a spanning surface for K.

Furthermore, X is a surface bundle over S^1 with monodromy of order pq. To show this, let S be the minimal spanning surface and consider $X - n(S)$.

Aside. To answer the question of a student, we define the Euler characteristic of an n–manifold.

Definition 24.6. We define $\chi(M^n)$ by taking a finite triangulation of M and setting $\chi(M) = \sum_{k=0}^{n}(-1)^k|T^{(k)}|$ where $|T^{(k)}|$ denotes the number of k-simplices in the image $\|T\|$.

Proposition 24.2 (1.12 in Hatcher)**.** *Suppose* $(M, \mathcal{F})$ *is compact, connected and Seifert fibred. Then* M *is irreducible or* M *is homeomorphic to one of* $S^2 \times S^1, S^2 \tilde{\times} S^1$ *or* $P^3 \# P^3$.

Proof. Suppose $S \subset M$ is an essential 2–sphere. Following the proof of Proposition 23.1 (1.11 in Hatcher) with surgery of essential surfaces replacing isotopy of essential spheres, we find an essential 2–sphere S' such that S' is vertical or horizontal. Since S' is not A^2, T^2, M^2 or K^2, we find S' must be horizontal.

1. If S' is non-separating, then $M - n(S')$ is homeomorphic to $S^2 \times I$. So $M \cong S^2 \times S^1$ or $S^2 \tilde{\times} S^1$.
2. If S' separates, then it is an exercise to show that $M \cong P^3 \# P^3$. □

Proposition 24.3 (1.13 in Hatcher)**.** *Let $(M, \mathcal{F})$ be as above. Then*

1. *every horizontal 2–sided surface is essential and*

2. *every vertical 2–sided surface is essential except for tori bounding fibred solid tori and boundary parallel annuli cutting off fibred solid tori.*

Proof. Suppose that D is a surgery disk or bigon for $S \subset M$.

1. Suppose S is horizontal. By the previous discussion, $M - n(S)$ is an I–bundle and D gives a surgery for $\partial_h(M - n(S))$. But the horizontal boundary of an I–bundle is always essential.

   Exercise 24.12. The horizontal boundary of an I–bundle is always essential.

2. Suppose S is vertical. So D gives a surgery in $M' \subset M - n(S)$ where M' is the component of $M - n(S)$ containing D. Suppose D is essential. Since $D \subset M'$ is essential, D must be vertical or horizontal, hence horizontal. Let $B = M'/S^1$.

   Exercise 24.13. Show that B is a disk with at most one orbifold point. Hint: use that $d \cdot \chi_{\text{orb}}(B) = \chi(D) = 1$.

   Thus M' is a solid torus. If D was a bigon, then, as $D \cap \partial M = D \cap \partial M'$ is a single arc, the fibring of M' is the trivial fibring, so $M' \cong V_{1,0}$. $\square$