MA4J2 Three Manifolds

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Lecture 25

Lemma 25.1 (1.14 in Hatcher)**.** *Let $A \subset (M, \mathcal{F})$ be an essential annulus. Then A can be properly isotoped to be vertical with respect to $\mathcal{F}$, possibly after changing $\mathcal{F}$ if M is $T \times I$, $T \tilde{\times} I$, $K \times I$ or $K \tilde{\times} I$.*

Proof. Since A is essential, it may be isotoped to be vertical or horizontal. Suppose A is horizontal. So $M - n(A)$ is an I–bundle with annuli as horizontal boundary components.

(i) If $M - n(A)$ is connected, then $M - n(A) \cong A \times I$. So

$$M = {}^{A \times I}\!\big/_{(x,1) \sim (\varphi(x), 0)} =: M_\varphi,$$

as in Figure 1.

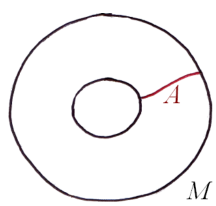

Figure 1: $M = (A \times I)/((x, 1) \sim (\varphi(x), 0))$.

But there are only four possibilities for φ, up to isotopy: the identity, reflections switching or preserving the boundary components, and the rotation given by composing these reflections. See Figure 2.

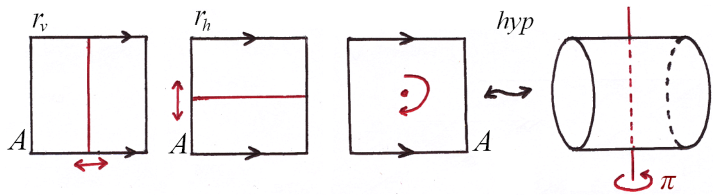

Figure 2: The three non-trivial possibilities for φ.

Exercise 25.1. Show that $\mathrm{MCG}(A) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here $\mathrm{MCG}(S)$ is the mapping class group of S, the group of homeomorphisms of S, up to isotopy.

These four maps give the four exceptions.

Exercise 25.2. Check this.

(ii) If $M - n(A)$ has two components, as in Figure 3, then $M - n(A) \cong \mathbb{M}^2 \tilde{\times} I \sqcup \mathbb{M}^2 \tilde{\times} I$.

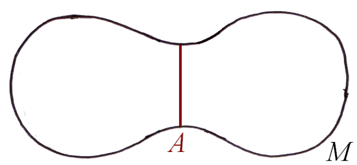

Figure 3: $M - n(A)$ may have two components.

Note that $\mathbb{M}^2 \tilde{\times} I$ is a cube with a pair of opposite faces glued by a π twist, shown in Figure 4.

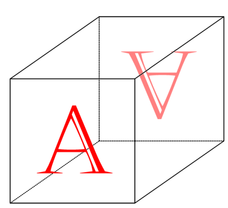

Figure 4: A picture of $\mathbb{M}^2 \tilde{\times} I$.

Exercise 25.3. Find the Möbius bands in this cube.

It is again an exercise to show that all four gluings give $K \tilde{\times} I$ with base orbifold $D^2(2, 2)$. □

Note. We have an exact sequence of groups: $S^1 \to K \tilde{\times} I \to D^2(2,2)$

$$\begin{array}{ccccccccc} \mathbf{1} & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(K^2) & \longrightarrow & D_\infty & \longrightarrow & \mathbf{1} \\ \mathbf{1} & \longrightarrow & \langle a^2 \rangle & \longrightarrow & \langle a, b \mid a^2 = b^2 \rangle & \longrightarrow & \langle a, b \mid a^2 = b^2 = 1 \rangle & \longrightarrow & \mathbf{1} \end{array}$$

coming from the long exact sequence for the Seifert fibering. See Theorem 4.41 page 276 of Hatcher's *Algebraic Topology* for more details.

Lemma 25.2 (1.15 in Hatcher)**.** *Let $(M, \mathcal{F})$ be as above. Then the slopes of $\mathcal{F}|\partial M$ are determined by M only, unless M is $V_{p,q}$ or one of the four exceptions above.*

Proof. If $\partial M = \varnothing$ then we have nothing to prove. If $B = M/S^1$ has no essential arcs, then $B = D^2(p)$.

Exercise 25.4. Check this.

Then $M \cong D \times S^1$ and we are done. So let $\alpha \subset B$ be an essential arc. See Figure 5.

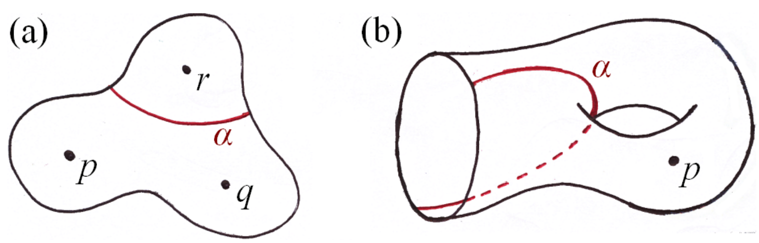

Figure 5: Two examples of essential arcs in (a) where $B = D^2(p, q, r)$ with $p, q, r > 1$, and (b) where $B = T^2 \# D^2(p)$.

Let $A \subset M$ be the vertical annulus above α. In this case:

(i) A is essential by Lemma 1.13 in Hatcher.

(ii) A is vertical in any fibering of M, with exceptions as above, by Lemma 1.14 in Hatcher.

So ∂A is determined by M alone, and we are done. □

Remark. Note that in the above we used the fact that solid Klein bottles are not Seifert fibered spaces.

Exercise 25.5. Show that the solid Klein bottle can be partitioned as a disjoint union of circles. Show, nonetheless, that the solid Klein bottle cannot be Seifert fibered.

Exercise 25.6. Show that $K \times I$ contains a solid Klein bottle, yet is still a Seifert fibered space.

Lemma 25.3 (1.16 in Hatcher)**.** *Suppose M is connected, compact, orientable, irreducible and atoroidal. Suppose $A \subset M$ is an essential annulus with ∂A contained in torus components of ∂M. Then M admits a Seifert fibering.*

Proof. Let M, A be as above. Let T be the components of ∂M meeting A. Let $N = N(A \cup T)$. So there are three cases:

(i) A meets two boundary components, T_1 and T_2, as in Figure 6.

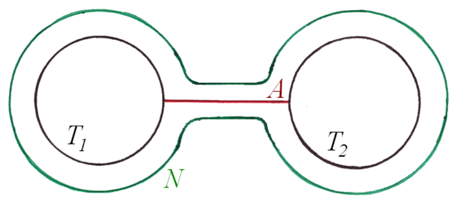

Figure 6: A meets two boundary components.

(ii) A meets a single boundary component without twisting, as shown in Figure 7.

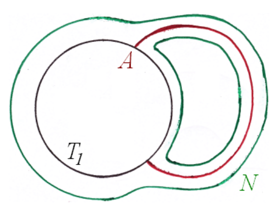

Figure 7: A meets a single boundary component without twisting.

(iii) A meets a single boundary component with a twist, as shown in Figure 8.

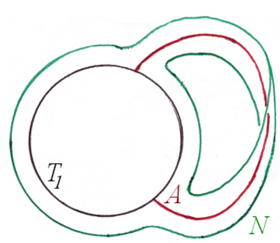

Figure 8: A meets a single boundary component with a twist.

Note. Note that Figures 6, 7 and 8 give a cross section of N. For example in Figure 6, the entirety of N is shown in Figure 9. Unfortunately the neighborhood N, in the third situation, does not embed in $\mathbb{R}^3$.

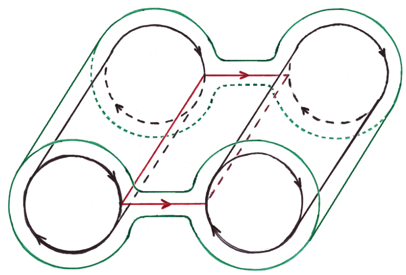

Figure 9: The whole of N in case (i), of which Figure 6 is a cross section. The front and back faces and edges are identifed.

Note that $N(A)$ and $N(T)$ are Seifert fibered, and we may glue these fibrings to get a fibering of N. Fix F, a component of $\partial N - \partial M$. In other words, a component of the *frontier* of N in M. Note that $F \cong \mathbb{T}^2$.

(i) Suppose that F compresses in M via a disk $(D, \partial D) \subset (M, F)$. Since A is essential we may arrange via an isotopy to have $A \cap D = \varnothing$. So we may assume that $D \cap N = \partial D$; thus F compresses to the "outside" of N. So F_D is a 2–sphere bounding a ball $B \subset M$. Note that $N \subset B$ is a contradiction as $\partial M \cap \partial N \neq \varnothing$. So $X = B \cup N(D)$ is a solid torus attached to F.

(ii) Suppose F is boundary parallel. Say $M - n(F)$ contains X, with $X \cong F \times I$ the parallelism. Since A is essential, we find that $X \cap N = F$, as $N \subset F$ leads to a contradiction.

So the fibering on N extends to a fibering on $N \cup X$. We do the same for all components of $\partial N - \partial M$. □

Exercise 25.7. Read the proof of Theorem 1.9 in Hatcher.

Lecture 26

We now state the Poincaré conjecture, proved by Perelman, following a program of Hamilton.

Poincaré Conjecture. *Suppose M^3 is closed and simply connected. Then M is homeomorphic to S^3.*

Recall that closed means that M is compact and $\partial M = \varnothing$. Simply connected means that M is connected and $\pi_1(M) = \{\mathbf{1}\}$. Note that the equivalent statement in dimension two follows from the classification of surfaces and the Seifert-van Kampen theorem. In dimensions greater than three, the conjecture was solved previously by (among others) Smale, Stallings, and for dimension four, Freedman.

Remark. Poincaré originally conjectured that if $H_1(M, \mathbb{Z}) = 0$ then $M = S^3$. He then gave a counterexample to this, called the *Poincaré homology sphere*. Let D be the dodecahedron and let $P = D/\sim$, where we glue opposite faces with a 1/10 right-handed twist, as in Figure 10.

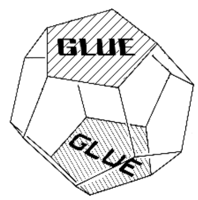

Figure 10: The Poincaré homology sphere. This diagram is adpated from one in The Shape of Space by J. Weeks.

Exercise 26.8. Let $\Gamma = \pi_1(P)$. Give a presentation of Γ and check that $\Gamma^{\text{ab}} = 0$.

Exercise 26.9. What if we use a 5/10 twist?

Remark. If we use a 3/10 twist we get the *Seifert-Weber dodecahedron space*. See Figure 11.

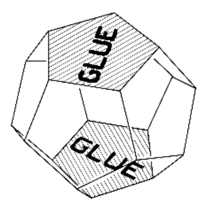

Figure 11: The Seifert-Weber dodecahedron space. This diagram is adapted from one in The Shape of Space by J. Weeks.

Definition 26.1. We say a knot $K \subset S^3$ is *spanned* by a surface $F \subset S^3$ if F is embedded and two-sided away from ∂F, and $\partial F = K$. In other words, the boundary of F wraps exactly once about K. See Figure 12a. Equivalently, $S \subset X_K$ is a *spanning surface* for K if it is two-sided, embedded, $|\partial S| = 1$ and the following holds. Let $N = N(K)$ and let $(D, \partial D) \subset (N, \partial N)$ be a meridian disk. Let $\mu = \partial D$. Then the transverse intersection $\mu \cap \partial S$ is a single point. See Figure 12b.

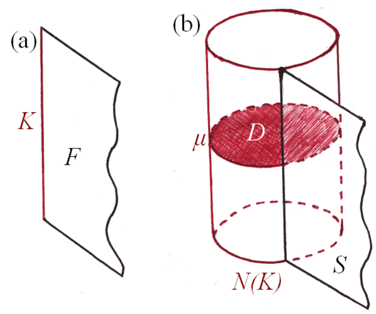

Figure 12: Diagrams of equivalent definitions of the spanning surface.

Recall that a knot K is the *unknot* if K is isotopic to a round circle.

Theorem 26.4. *Suppose $K \subset S^3$ is a knot. The following are equivalent:*

(i) K *is the unknot.*

(ii) K *is spanned by a disk* E.

(iii) $X_K = S^3 - n(K)$ *is a solid torus.*

(iv) $\pi_1(X_K) \cong \mathbb{Z}$.

See Figure 13.

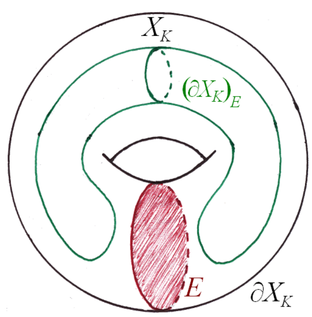

Figure 13: Illustration of Theorem 26.4.

Proof.

(i) $\Longrightarrow$ (ii) Use ambient isotopy.

(ii) $\Longrightarrow$ (iii) Use irreducibility of X_K and the fact that $(\partial X_K)_E \cong S^2$. Note that $E \subset X_K$ is essential as $\partial E \cap \mu$ is a point. So if $(\partial X_K)_E$ bounds a 3–ball B, we have $B \cup N(E) \cong E \times S^1$ is a solid torus.

(iii) $\Longrightarrow$ (i) This follows from Exercises 2.2 and 6.6.

(iii) $\Longrightarrow$ (iv) Since $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, we have $\pi_1(X_K) \cong \pi_1(S^1) = \mathbb{Z}$.

(iv) $\Longrightarrow$ (iii) We must show that if M is irreducible, $\partial M = \mathbb{T}^2$ and $\pi_1(M) \cong \mathbb{Z}$, then $M \cong D \times S^1$. This requires Dehn's Lemma. □

Exercise 26.10. Deduce (iv) $\Longrightarrow$ (iii) from the following lemma.

Dehn's Lemma (Papakyriakopoulos, 1957)**.** *Suppose $\alpha \subset \partial M$ is a simple closed curve, bounding a singular disk in M. Then α bounds an embedded disk in M.*

Loop Theorem. *Suppose F is a component of ∂M, and $i_*\colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential simple closed curve $\alpha \subset F$ such that $[\alpha] = \mathbf{1} \in \pi_1(M)$.*

This leads nicely to the following conjecture.

Simple Loop Conjecture. *If $i\colon F \looparrowright M$ is a two-sided map, and i_* is not injective, then there is an essential simple loop in the kernel.*

This has been proved by Gabai if M is a surface, and by Hass if M is Seifert fibered.

Exercise 26.11. Prove the simple loop conjecture when F is two-sided and properly embedded in M.

Lecture 27

Disk Theorem. *Suppose that $F \subset \partial M$ is a component, and $i_*\colon \pi_1(F) \to \pi_1(M)$ is not injective. Then there is an essential disk $(D, \partial D) \subset (M, F)$.*

Exercise 27.12. Show that the Disk Theorem is implied by the Loop Theorem and Dehn's Lemma.

The Disk Theorem is the first "promotion" theorem, among many others. For example we have the following:

Sphere Theorem. *Suppose M is an orientable 3–manifold with $\pi_2(M)$ non-trivial. Then there is an embedded 2–sphere $S \subset M$ such that $[S] \neq \mathbf{1} \in \pi_2(M)$.*

In general we assume that there is an essential map $(F, \partial F) \looparrowright (M, \partial M)$. The corresponding promotion theorem gives us an embedding. For example, F could be a disk or sphere (due to Papakyriakopoulos), a projective plane (due to Epstein), an annulus or torus, or indeed any F with $\chi(F) \geq 0$.

We now discuss hierarchies. Suppose that $M_0 = M$, suppose that $S_i \subset M_i$ is a properly embedded two-sided surface, and define:

$$M_{i+1} := M_i - n(S_i).$$

So we have a sequence of manifolds:

$$M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} M_n.$$

Definition 27.2. Call a sequence $\{M_i, S_i\}$ a *partial hierarchy* if every S_i is essential in M_i.

Note. Some authors only require S_i to be incompressible.

The following example demonstrates why we require the S_i to be essential.

Example 27.1. Take annuli in V_2, the genus 2 handlebody, as in thr right hand side of Figure 14, and glue them to give $M_0 \cong V_2$. Let S_0 be the single annulus given by the image of the two annuli under the gluing map. Then cutting along S_0 gives $M_1 \cong V_2$, so we could continue the process indefinitely.

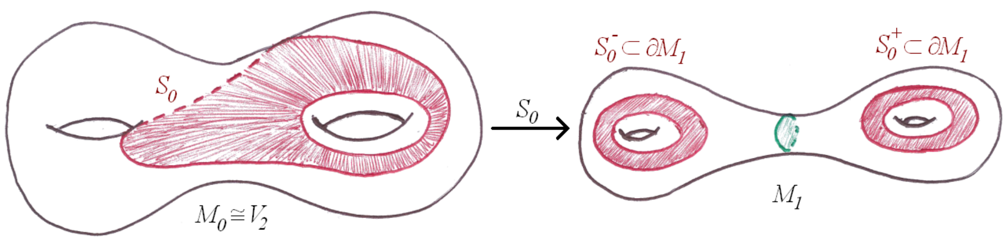

Figure 14: Note that S_0 is inside M_0, not on the boundary (although $\partial S_0 \subset \partial M_0$).

Equivalently, one can think of V_2 as $\big(T^2 - n(\big(B^2\big)\big) \times I$, as in Figure 15.

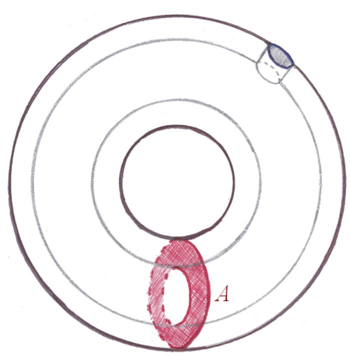

Figure 15: Another way to look at V_2.

Cut along A to get the pair of pants $\times I$, as in Figure 16.

Figure 16: Let $F = T - \text{int}(D)$ be a once-holed torus. Let G be a pair of pants. Cutting $F \times I$ along a vertical annulus gives a copy of $G \times I$. As $F \times I \cong G \times I$ this could lead to an infinite hierarchy, were we to allow non-essential surfaces.

Definition 27.3. If M_n is a collection of 3–balls, then the partial hierarchy is simply called a *hierarchy.*

Example 27.2. Let $M_0 = \mathbb{T}^3$, thought of as the unit cube in $\mathbb{R}^3$ with face pairings. Let $S_0 \subset M_0$ be the image of the xy–plane, so $S_0 \cong T^2$. Then $M_1 \cong T \times I$. Let S_1 be the image of the yz–plane, so $S_1 \cong A^2$, and $M_2 \cong D \times S^1$. Let S_2 be the image of the zx–plane, a disk. Then $M_3 \cong B^3$. See Figure 17.

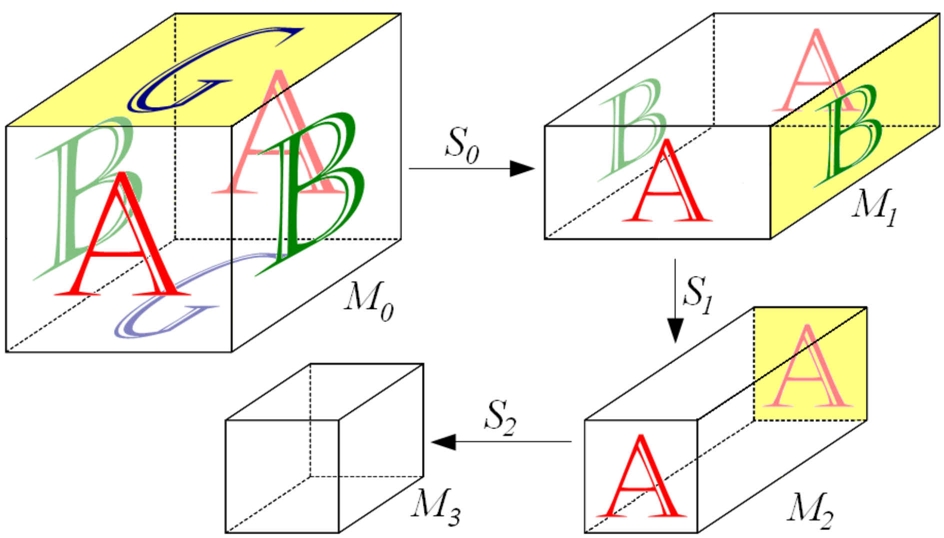

Figure 17: A hierarchy of length three for the three-torus.

Example 27.3. Let $M_0 = X_K$, where K is the (p, q)–torus knot, as shown in Figure 18, and let $S_0 = A$, the unique essential annulus. Then $X_K \xrightarrow{A} V_K \sqcup W_K = M_1$. Now letting S_1 be a pair of meridian disks, one in each of V_K and W_K, we find that $M_2 \cong B_1^3 \sqcup B_2^3$. See Figures 19 and 20.

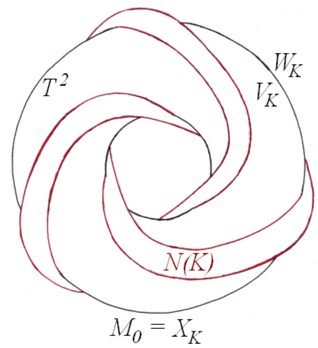

Figure 18: The (p, q)–torus knot complement, M_0.

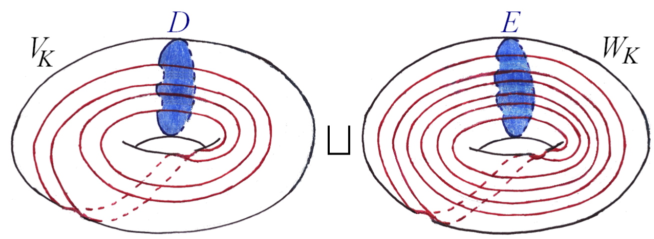

Figure 19: Compressing disks for V_K and W_K.

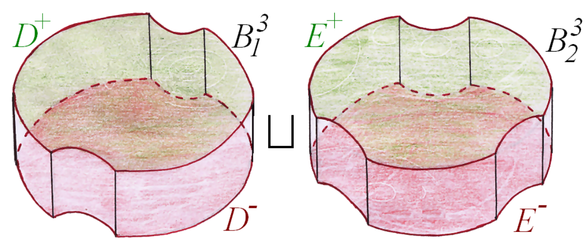

Figure 20: The final stage of the heirarchy.

Definition 27.4. If M is compact, orientable and irreducible, and $S \subset M$ is properly embedded, two-sided and essential, then M is called *Haken*.

Theorem 27.5. *If M is compact, orientable, irreducible and $\partial M \neq \varnothing$, then either M is a 3–ball or M is Haken.*

This theorem is implied by the following:

Theorem 27.6. *If M is compact, orientable and irreducible, and if*

$$\text{rank}(H_1(M, \mathbb{Z})) \geq 1$$

then M is Haken.