1 Lecture 1

1.1 Manifolds

A goal of Topology is to classify closed (that is, compact, connected, without boundary) manifolds.

Definition. A (Topological) **Manifold** M^n is a Hausdorff, 2nd countable topological space in which every point $x \in M^n$ has a neighbourhood $U \subseteq M^n$ homeomorphic to $\mathbb{R}^n$, for a manifold without boundary, or homeomorphic to $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n \geq 0\}$, for a manifold with boundary.

1.2 Some Examples

1.2.1 $n = 0$

$\mathbb{R}^0$ - a single point.

1.2.2 $n = 1$

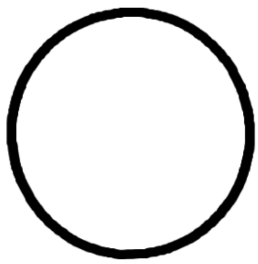


Figure 1: $\mathbb{S}^1$ - the circle.

Definition. $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

$\mathbb{R}^1$ is not compact. $\mathbb{I} = [0, 1] \subseteq \mathbb{R}$ is not closed. (the boundary of $\mathbb{I}$, $\partial\mathbb{I} = 0, 1$, see Figure 2)

Figure 2: The semicircle is homeomorphic to $\mathbb{S}^1$.

Definition. A map $f : M \mapsto N$ is a homeomorphism if f is bijective, continuous and has a continuous inverse.

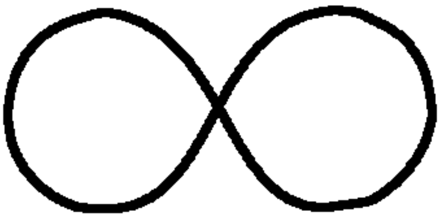
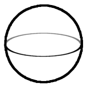
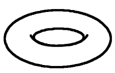
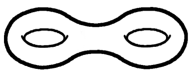
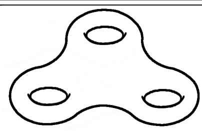


Figure 3: A figure of 8.

The figure of eight curve (see Figure 3) is not a manifold.
There is no neighbourhood of the centre point which is homeomorphic to $\mathbb{R}$.

Exercise 1. *(Hard)*
Prove: $\mathbb{S}^1, [0,1], [0,\infty], \mathbb{R}$ are the only connected 1-manifolds.

1.2.3 $n = 2$

genus(g)	0	1	2	3
S_g				
N_{g+1}	$\mathbb{P}$	$\mathbb{K}$	N_3	N_4

Here, S_g is the compact, connected orientable surface with g handles. We

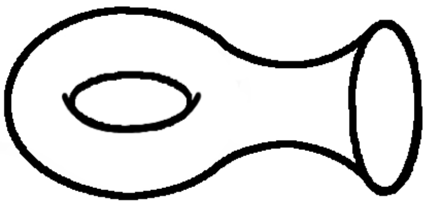


Figure 4: A handle.

get a handle (Figure 4) by removing a disk $\mathbb{D}^2$ from T^2. See Figure 5.

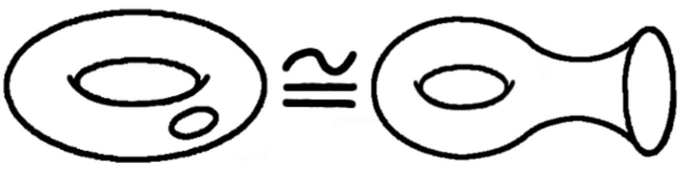
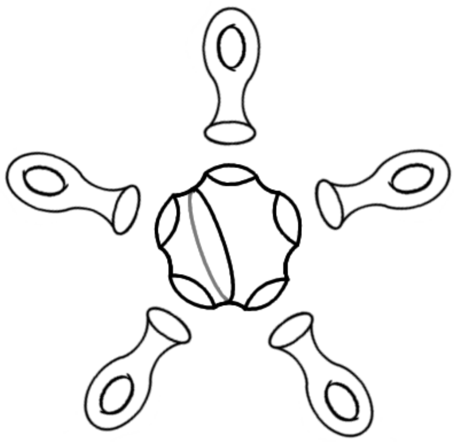


Figure 5: Creating a handle.

So we may draw S_g as a sphere with g disks removed.



Example. $g = 5$

Example. N_c is the closed, non-orientable surface with c cross-caps.
We get a cross-cap by taking S^2, cutting out c disks and glueing on c copies of the mobius band. See Figure 6.

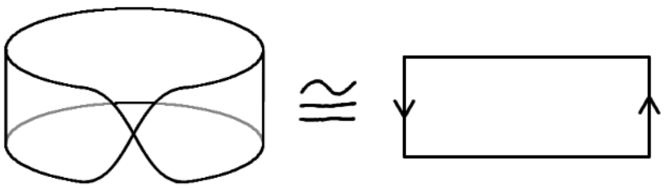


Figure 6: $\mathbb{M}^2$.

Note: $\partial\mathbb{M}^2 = S^1$
Note: $\mathbb{A}^2 = annulus = S^1 \times \mathbb{I}$ has $\partial\mathbb{A}^2 - S^1 \sqcup S^1$
$\mathbb{A}^2 \cong \{x \in \mathbb{R}^2 : 1 < \|x\| \leq 2\}$. See Figure 7.

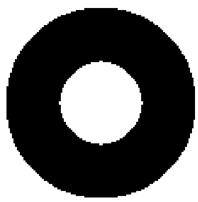


Figure 7: Annulus.

1.2.4 $n = 3$

This is the topic of this class.

Recent work of Pevelman, Hamilton, Thurston (also Casson, Rubenstein, Manning...) proves that the homeomorphism problem in dimension *is* decidable.

In dimensions 3 and below, the phrases "up to homeomorphism" and "up to diffeomorphism" are *equivalent* [Moies]. This is not true in higher dimensions.

1.2.5 $n = 4$

Impossible in the sense of Gödel & Turing (Markov, 1958).

1.3 Possible structures of manifolds

There are many possible structures we can put on manifolds.

- Topological - TOP
- Piecewise linear - PL
- DIFF - smooth
- $\mathbb{C}^\omega$ - real analytic
- $\mathbb{C}$ - analytic (in even dimensions)
- Geometric ← **this course**

Definition. $\mathbb{P}^n = \frac{\mathbb{S}^n}{\{\pm 1d\}} = \frac{\mathbb{S}^n}{x \sim (-x)}$

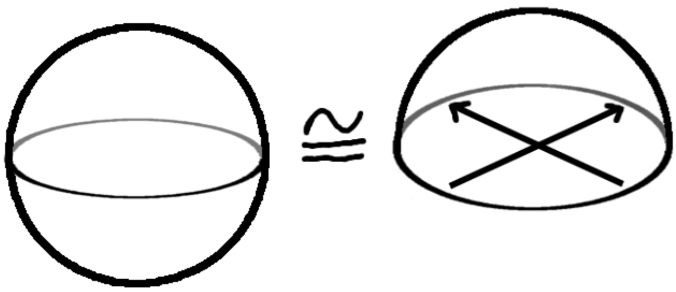


Figure 8: The Projective Plane.

See Figure 8

- $\mathbb{T}^0$ is a point
- $\mathbb{T}^1 \cong S^1$
- $\mathbb{T}^2$ is the torus we are most familiar with, see Figure 9
- $\mathbb{T}^3 \cong \mathbb{I}^3 \leftarrow unitcube = [0,1]^3$ Glue opposite sides by translation. See Figure 10.

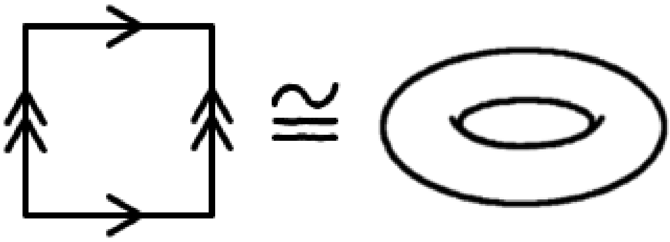


Figure 9: $\mathbb{T}^2$

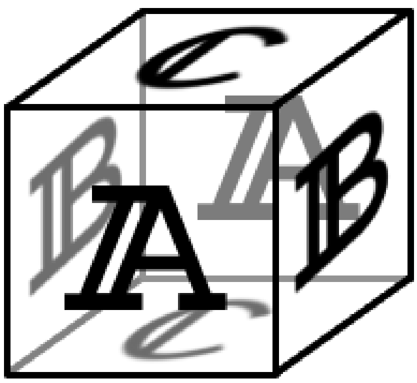


Figure 10: $\mathbb{T}^3$