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10 Lecture 10

In this lecture, we aim at classifying geodesics in the sphere $\mathbb{S}^2$ and in the hyperbolic plane $\mathbb{H}^2$.

10.1 Geodesics in $\mathbb{S}^2$

We would like to classify geodesics in the sphere. Here is an outline to prove that they are great circles.

Step 1 (Finding Isometries.). *Firstly, we show that $O(3)$ is a subgroup of Isom$(\mathbb{S}^2)$. To do this, we have to show that $O(3)$ preserves S^2 and to observe that $O(3)$ preserves the length element ds, where $ds^2 = dx^2 + dy^2 + dz^2$, thus $O(3) \leq$ Isom$(\mathbb{E}^3)$. If α is an element in $O(3)$, then it preserves length in $\mathbb{E}^3$, thus in particular it leaves invariant the integral $\int_\gamma ds$ (see Figure 1).*

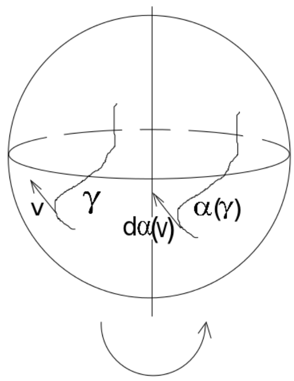

Figure 1: Moving paths on $\mathbb{S}^2$ by a rotation α.

Step 2 (Choosing coordinates.). *Rewrite ds in spherical coordinates. Introducing new parameters (r, θ, ϕ) to denote the distance from the origin, the angle out of the xz plane and the angle from the z-axis, respectively, we can write relations between Euclidean and spherical coordinates as follows (see Figure 2):*

$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Computing dx, dy and dz in terms of dr, $d\theta$ and $d\phi$ we get

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi \; d\theta^2. \tag{1}$$

Along the sphere $r \equiv 1$ and so $dr \equiv 0$. Thus equation (1) becomes

$$dx^2 + dy^2 + dz^2 = d\phi^2 + \sin^2\phi \; d\theta^2.$$

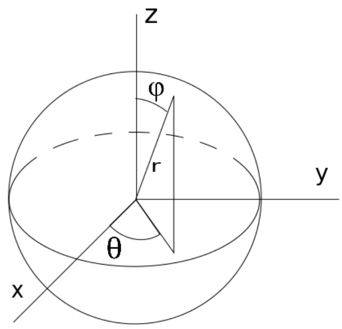

Figure 2: Spherical coordinates.

Step 3 (Projecting to geodesic arcs.). *We are ready to show that arcs of great circles running from the south pole to the north pole of the sphere are geodesics. Let* $\gamma\colon [0,1] \to \mathbb{S}^2$ *be a path connecting the south pole* $\mathfrak{S} = (0,0,-1)$ *to a point of* $S^2 \cap \{xz \text{ plane}\}$ *with* $x > 0$ *and assume* γ *to be injective. The idea is to project* γ *to the path* $\bar{\gamma}$ *as shown in Figure 3, in order to reduce the length.*

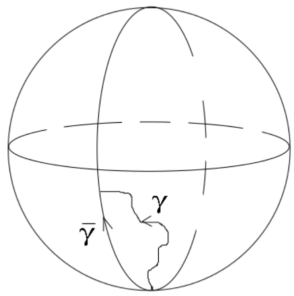

Figure 3: Projecting on $\bar{\gamma}$.

We have:

$$l(\gamma) = \int_\gamma \sqrt{d\phi^2 + \sin^2\phi \; d\theta^2} \geq \int_\gamma |d\phi| = \int_{\bar{\gamma}} |d\phi| = l(\bar{\gamma})$$

so that $\bar{\gamma}$ *is a geodesic.*

Step 4 (Moving geodesics around.). *Using Step 1, conclude that any great circle contained in an open hemisphere is a shortest path.*

Step 5 (Nothing more to look for.). *Finish the proof by showing that there are no more isometries of the sphere. That is, show* $O(3) = Isom(\mathbb{S}^2)$.

10.2 Geodesics in $\mathbb{H}^2$

Let $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 | y > 0\}$ denote the upper half plane equipped with the element of length $ds_{\mathbb{H}} = \frac{ds}{y}$. We will switch between real and complex notation if needed, writing (x,y) for the complex number $z = x + iy$ and vice versa. Figure 4 gives us already a first idea of the behaviour of the metric.

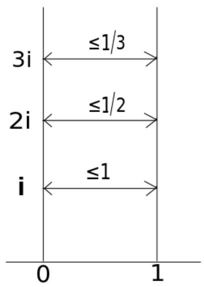

Figure 4: Hyperbolic metric on the upper halp plane.

Note that the inequalities shown in the picture cannot be turned into equalities. This is because horizontal lines are not geodesics. We wish to classify geodesics in $\mathbb{H}^2$ in a similar fashion as what we have done for the sphere. However, we will proceed in a slightly different way, so that we will not cover every step and not in the same order. Let's start with:

Step 6 (Projecting to geodesic arcs)**.** *As before, we have to look at the metric and find "nice" paths to project to. The length element squared is*

$$ds_{\mathbb{H}}^2 = \frac{ds^2}{y^2} = \frac{dx^2 + dy^2}{y^2}.$$

We try making $dx = 0$, that is considering vertical paths. Let $\gamma\colon [0,1] \to \mathbb{H}^2$ be any path (again, γ is assumed to be an injective map) that starts at $i = (0,1)$ and ends at Ri for some $R > 1$. If $\bar{\gamma}$ is the path shown in Figure 5 having the same endpoints as γ,
then:

$$l(\gamma) = \int_\gamma ds_{\mathbb{H}} = \int_\gamma \frac{\sqrt{dx^2 + dy^2}}{y} \geq \int_{\bar{\gamma}} \frac{|dy|}{y} = l(\bar{\gamma}).$$

Hence the open semi-line $i\mathbb{R}_+$ is a geodesic.

Step 7 (Finding isometries.)**.** *Now that we have found one geodesic, we want to find some elements in $Isom(\mathbb{H}^2)$ that will allow us to move it around, getting all the other geodesics. Let $T_a\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the translation $T_a(z) = z + a$ (or, equivalently, $T_a(x,y) = (x+a,y)$) where $a \in \mathbb{R}$. Then T_a is an isometry. For its derivative is equal to the identity, so it preserves lengths. This tells us already that all vertical lines are geodesics.*

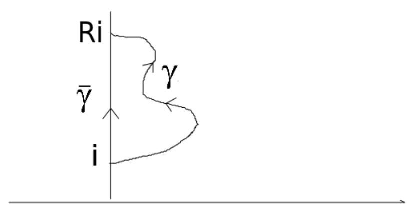

Figure 5: Projection in $\mathbb{H}^2$.

Consider now the function $S_\lambda(x, y) = (x, \lambda y)$; is it an isometry? Let's check how its derivative acts on the length element: set $(u, v) = S_\lambda(x, y)$ so that $(du, dv) = (dx, \lambda dy)$ and compute

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + \lambda^2 dy^2}{\lambda^2 y^2} \neq ds^2_{\mathbb{H}}.$$

So S_λ is not an isometry. We can modify it in a suitable way. Take instead $\tilde{S}_\lambda(x, y) = (\lambda x, \lambda y)$ (in complex notation $\tilde{S}_\lambda(z) = \lambda z$). The same computation leads to:

$$\frac{du^2 + dv^2}{v^2} = \frac{\lambda^2 dx^2 + \lambda^2 dy^2}{\lambda^2 y^2} = ds^2_{\mathbb{H}}$$

as desired. So we have found a family of isometries $\tilde{S}_\lambda$. All the isometries we have found so far move vertical lines to vertical lines, so we are not done yet.

10.3 Further remarks

Now that we know that vertical lines are geodesic with respect to the hyperbolic metric, we are ready to find new isometries and, consequently, new geodesics. We have seen that both translations and dilations by real numbers are isometries. Also, they move vertical lines to vertical lines.
Observe the reflection $R\colon z \mapsto -\bar{z}$ about the imaginary semi-line is an isometry. So is reflection in any vertical line, as this can be obtained by composition as $R_k = T_k \circ R \circ T_{-k}$, giving the map $z \mapsto 2k - \bar{z}$. Unfortunately, the class of reflections about vertical lines still preserves vertical lines; moreover, these reflections are orientation reversing. Let's put them aside for a moment, and look at the map $I\colon z \mapsto 1/\bar{z}$, namely inversion with respect to the unit circle. Writing this function in real coordinates we get

$$(x, y) \mapsto \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Computing the differential we can check that the map is actually an isometry. For, switch to polar coordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$ so that $dx = dr\cos\theta - r\sin\theta d\theta$ and $dy = dr\sin\theta + r\cos\theta d\theta$. the length element can be thus expressed as

$$\frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\theta^2}{r^2 \sin^2\theta}.$$

Writing the inversion as $(x, y) \mapsto (r^{-1}\cos\theta, r^{-1}\sin\theta) = (u, v)$, we have to compute the new length element given by $(du^2 + dv^2)/v^2$. We have

$$du = \frac{-r\sin\theta d\theta - dr\cos\theta}{r^2}$$

and

$$dv = \frac{\cos\theta d\theta r - dr\sin\theta}{r^2}.$$

Therefore the length element after inverting by I is

$$\frac{du^2 + dv^2}{v^2} = \frac{dr^2 + r^2 d\theta^2}{r^2\sin^2\theta} = \frac{dx^2 + dy^2}{y^2}$$

thus we proved that the inversion is actually an isometry. As before, conveniently conjugating I with dilations and translations gives rise to inversions about circles with any centre on the real line and any positive radius. Note that all of these inversions are again orientation reversing: in particular, the composition of an inversion with a reflection in a line is an orientation preserving isometry. For instance, the composition $I \circ R$ is the map $z \mapsto -1/z$. We also note that a vertical line is sent to a semi-circle orthogonally meeting the line $\{z \mid \Im z = 0\}$. Thus, we have found that vertical lines and semi-circle with centre on the real line are geodesics in $\mathbb{H}^2$. Another way of thinking at them is consider vertical lines as circles passing to the point at infinity.
It is possible to prove that the (orientation preserving) isometries that we have found generate the whole group $\mathrm{Isom}^+(\mathbb{H}^2)$, the other isometries being obtainable by compositions. More precisely, the group of orientation preserving isometries of the hyperbolic plane can be identified with the group

$$\mathbb{P}SL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm I$$

under the monomorphism

$$\mathbb{P}SL(2,\mathbb{R}) \to \mathrm{Isom}^+(\mathbb{H}^2), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left[z \mapsto \frac{az+b}{cz+d} \right]$$

Note that we take the projectivised group in order to avoid the ambiguity in choosing a sign for the matrix. Finally, we observe that for any two distinct points $p \neq q \in \mathbb{H}^2$ there exists a unique geodesic passing through them (i.e. $\mathbb{H}^2$ is a geodesic space).