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12 Lecture 12

Question: What was the definition of discrete?
Answer: G acting on X acts discretely if and only if G acts properly discontinuously.
OR
$G < Isom(X)$ is discrete as a subgroup if and only if $G \subset Isom(X)$ is discrete as a subset.
We may equip $Isom(X)$ with various natural topologies; luckily for us, these all agree!

Example 12.1. We could use the *compact-open topology*; suppose $K, U \subset X$ are compact and open respectively.
Then the set $V_{K,U} = \{g \in Isom(X) : g(K) \subset (U)\}$ is open. These form a sub-basis for the topology on $Isom(X)$.
In fact, the compact-open topology makes sense for $Homeo(X) = \{f\colon X \to X : f homeomorphism\}$ and $Isom(X)$ gets the subspace topology.

Back to Orbifolds
Suppose F is a 2-dimensional orbifold and $x \in F$ (that is, $(U, x) \subset (\Re^2_+/G, [0])$) where $G < O(2)$ is discrete, and so finite.
We call $G = G_x$ the local group of x in F.
ManifoldsFigure1.png

Definition 12.2. A map of orbifolds $\rho\colon E \to F$ is an *orbifold covering map* if for every $x \in F$ there is a sufficiently small neighbourhood $U \ni x$ such that $\forall y \in \rho^{-1}(x)$, the component $V \subset \rho^{-1}(U)$ is sufficiently small and the local group $G_y < G_x$ so that the diagram below commutes:
COMMUTATIVE DIAGRAM

Example 12.3. FIGURE S(2,2,2)

Definition 12.4. If $\rho\colon E \to F$ is an orbifold cover, then if x, U, y, V are defined as before, define $deg(\rho|V) = [G_x : G_y]$ (the index of G_y in G_x.
Define also
$deg(\rho, x) = \sum_{y \in \rho^{-1}(x)} [G_x : G_y]$

Exercise 12.5. If F is connected then $deg(\rho, x)$ is independent of x.
FIGURE
$deg(\rho, green) = 4 + 4 \times 2 = 12$
$deg(\rho, blue) = 6 + 2 \times 3 = 12$
$deg(\rho, red) = 6 + 3 + 2 + 1 = 12$

Theorem 12.6. *If $\rho\colon E \to F$ is an orbifold cvering and F is connected, then* $\chi^{orb}(E) = deg(\rho) \times \chi^{orb}(F)$

Proof
Exercise.

Remark 12.7. In $\mathbf{E}^2$ geometry, this is not so useful (Why?). But, it is extremely useful for $\mathbf{S}^2$ and $\mathbf{H}^2$ geometry.

Exercise 12.8. Classify all degree 2 coverings between the frieze orbifolds.

Challenge
Do the same for wallpaper orbifolds.

Definition 12.9. An orbifold F is *good* if there is an orbifold covering $\rho\colon E \to F$ where E is a surface. Else F is *bad*.

To think about: Are there any bad orbifolds?