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13 Lecture 13

We assume that if $(X, \mathrm{Isom}(X))$ is a geometry then the space X that we are considering is a manifold equipped with a metric (i.e. a Riemannian manifold), in addition to being a simply connected, homogeneous and complete space.

Exercise 13.1. Find a metric space X which is homogeneous, connected and simply connected but which is not a manifold.

Remark 13.2. Henceforth all geometries mentioned in the lectures are considered to have underlying space X which is a manifold.

Let us discuss possible solutions to the exercise above,

Question 1. What about the *tripod*?

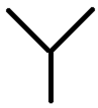

Figure 1

Answer: This is not homogeneous.

Question 2. What about the *long line*? [See Wikipedia]
Answer: The long line is not metrizable.

Remark 13.3. If a space is locally Euclidean at one point and if the space is homogeneous then the whole space is locally Euclidean.

Example 13.4. Consider the *Sierpinski Gasket*, shown in Figure 2.

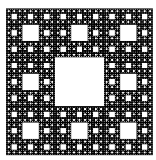

Figure 2: The Sierpinski Gasket is constructed by considering the square as the union of 9 equally sized squares. Remove the central one and repeating this process with each smaller and so on.

However the *Sierpinski Gasket* is not simply connected. So it does not answer our question. However, what if we take the universal cover $\widetilde{S}$ and ask if $\widetilde{S}$ is homogeneous?

Remark 13.5. We could try this procedure with the *Sierpinski triangle*, shown in Figure 3.

Claim 1. T (and so $\tilde{T}$) are <u>not</u> homogeneous.

Question 3. Does $\widetilde{S}$ exist?

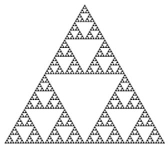

Figure 3: Constructed similarly to the *Gasket*. Subdivide the triangle into four equally sized triangles and remove the centre.

Remark 13.6. Because $\pi_1(S)$ is uncountable, I am not sure what $\widetilde{S}$ is. See Hatcher/Intro to Topology for the requirements of the base space to have a universal cover. These are delicate questions of point-set topology.

Recall 1. An orbifold is *good* if there is an orbifold cover $\rho\colon E \to F$, where E is in fact a surface. Otherwise F is *bad*.

Theorem 13.7 (Thurston)**.** *There are only the following bad, compact orbifolds in dimension 2:*

$\underline{S^2(p),\ p \neq 1}$

Figure 4: The Teardrop

$\underline{S^2(p,q),\ p \neq q}$

Figure 5: The Spindle

$\underline{D^2(\bar{p}),\ p \neq 1}$

Figure 6: The Monogon

$\underline{D^2(\bar{p},\bar{q}),\ p \neq q}$

Figure 7: The Bigon

Theorem 13.8. *All the other 2-orbifolds are good.*

This is Theoerem 2.3 [page 425] of Scott's notes, there are various proofs of this theorem.

Exercise 13.9. Prove the orbifolds on the list above are bad.

Remark 13.10. These are 2-fold covers: $S^2 \to D^2(\bar{p})$ and $S^2(p,q) \to D^2(\bar{p},\bar{q})$.

Exercise 13.11. If $\rho\colon E \to F$ is an orbifold cover and if F is good, then E is good.

Remark 13.12. The converse, If E is good then F is good is trivial. That is, if E is good then there exists an orbifold cover $\sigma\colon G \to E$ such that G is a surface. Then [*Check!*] $\rho \circ \sigma\colon G \to F$ is an orbifold cover.

Theorem 13.13 (Theorem 2.4 of Scott)**.** *If F is a connected, good 2-orbifold, without regular boundary points then F admits a geometry modelled on one of S^2, $\mathbb{E}^2$, $\mathbb{H}^2$. That is, there is a discrete $G < \mathrm{Isom}(X)$ such that $F \cong X/G$ (isomorphic as orbifolds).*

So let us consider, $S^2(2,2,2)$:

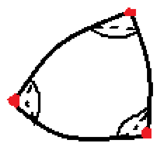

Figure 8: $S^2(2,2,2)$

Claim: This is a 4-fold cover by S^2.

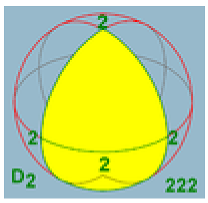

Figure 9: The three rotations of the 2-sphere

Claim 2. The diagram gives the quotient as $S^2(2,2,2)$.

Exercise 13.14. Verify this by showing the deck group is generated by the three 180° rotations about the x, y and z axes.

Remark 13.15. On Theorem 2.4: As F is good (Proposition), there is a <u>finite</u>, regular cover $\rho\colon E \to F$ by a surface E, then $\mathrm{Deck}(\rho)$ acts on E by homeomorphisms. $E/\mathrm{Deck}(\rho) \cong F$ and then apply the uniformisation theorem. The definition of a good orbifold doesn't mention a finite cover, so to prove Theorem 2.4 via this method, we need to do a little bit of work to establish the existence of a finite cover.

Definition 13.16. If $\rho\colon E \to F$ is an orbifold cover, E is a surface *and* $\pi_1(E) = 1$ then we call E a *universal cover* of F.

Definition 13.17. In this case, we write $\pi_1^{orb}(F) = \mathrm{Deck}(\rho\colon E \to F)$

Example 13.18. Consider the exponential map from $\mathbb{R}$ to S^1, $\exp\colon \mathbb{R} \to S^1$, $\theta \mapsto e^{i\theta}$ then $\pi_1(S^1) \cong \mathrm{Deck}(\exp) \cong \mathbb{Z}$

Example 13.19. Consider the covering map from $\mathbb{R}^2$ to the 2-torus, $p\colon \mathbb{R}^2 \to \mathbb{T}^2$, $(\theta, \phi) \mapsto (e^{i\theta}, e^{i\phi})$ then $\pi_1(\mathbb{T}^2) \cong Deck(p) \cong \mathbb{Z}^2$