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15 Lecture 15

We have several more examples of computing orbifold fundamental groups $\pi_1^{orb}(F)$.

Example 15.1. We compute the group for $F = D^2(\overline{3}, \overline{3}, \overline{3})$.

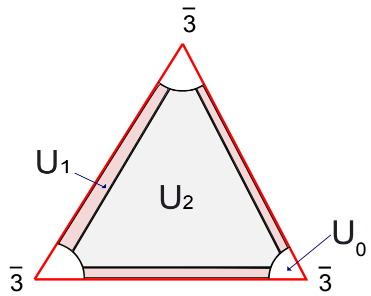


Figure 1: The orbifold decomposed into pieces.

Let U_0 be a small regular neighborhood of the corner reflectors. Let U_1 be an even smaller regular neighborhood of the remains of the mirror boundary.

Let $U_2 = \overline{F - (U_1 \cup U_0)}$. Notice that $\pi_1^{orb}(U_2) = \{\mathbf{1}\}$. The components of U_1 each have $\pi_1^{orb} \cong \mathbb{Z}_2$. For U_0, each component has $\pi_1^{orb} \cong D_6 \cong \left\langle a, b \mid a^2, b^2, (ab)^3 \right\rangle$. Recall that D_6 is symmetry group of equilateral triangle.

Thus: $U_1 \cup U_2$ is the *half mirrored hexagon, H* . The orbifold fundamental

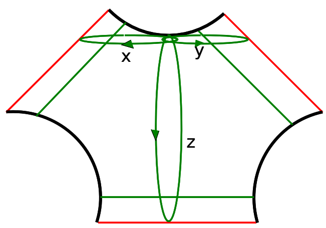


Figure 2: The half mirrored hexagon.

group of H is $\pi_1^{orb}(H)$ equal to

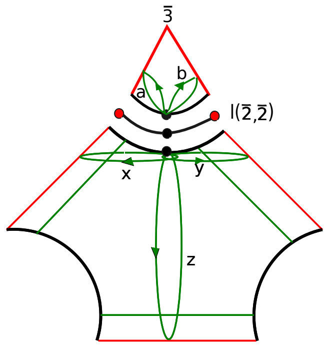
$$\mathbb{Z}_2 \; * \; \mathbb{Z}_2 \; * \; \mathbb{Z}_2 \; \cong \; \langle x, y, z \mid x^2, y^2, z^2 \rangle .$$

Let A be a component of U_0. Let $H = U_1 \cup U_2$. We must glue A to H along the relevant copy of $I(\overline{2}, \overline{2})$. van Kampen Theorem tells us that $\pi_1^{orb}\left(A \bigcup_{I(\overline{2},\overline{2})} (H)\right)$ is the group

$$(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2) *_{D_\infty} D_6.$$

If $D_\infty = \langle \alpha, \beta \mid \alpha^2, \beta^2 \rangle$ then we have

$$\begin{array}{rcll} f: & I(\overline{2}, \overline{2}) & \longrightarrow & \mathrm{A}, \\ g: & I(\overline{2}, \overline{2}) & \longrightarrow & H, \end{array}$$



$$f_*: \begin{array}{l} \alpha \longmapsto a, \\ \beta \longmapsto b, \end{array}$$
$$g_*: \begin{array}{l} \alpha \longmapsto b, \\ \beta \longmapsto y. \end{array}$$

Thus:

$$\pi_1^{orb}(A \cup H) = \left\langle x, y, z, a, b \mid x^2, y^2, z^2, a^2, b^2, (ab)^3, ax^{-1}, by^{-1} \right\rangle.$$

Remark 15.2. There are more relations; for example, $f_*(\alpha\beta)\, g_*\left(\beta^{-1}\alpha^{-1}\right)$ giving $aby^{-1}x^{-1}$. These follow from the relations ax^{-1} and by^{-1}.

Exercise 15.3. Check this.

This group can now be written as

$$\pi_1^{orb}(A \cup H) \cong \left\langle x, y, z \mid x^2, y^2, z^2, (xy)^3 \right\rangle.$$

Glue on the remaining two corners to get

$$\pi_1^{orb}\left(\mathbb{D}^2\left(\overline{3}, \overline{3}, \overline{3}\right)\right) = \left\langle x, y, z \mid x^2, y^2, z^2, (xy)^3, (yz)^3, (zx)^3 \right\rangle.$$

Exercise 15.4. We now carry out the same exercise for $F = S^2(3,3,3)$ Cut off neighborhoods of the 3 corners. As in (Fig: 3) we have;

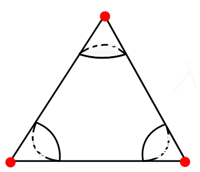


Figure 3: $S^2(3,3,3)$ is the 2-sphere with 3 cone points.

In (Fig: 4) X is $\mathbb{D}^2 - \{2\ disks\}$. A, B and $C$are $\mathbb{R}^3/\mathbb{Z}_3$. For each of these the local group is $C_3 \cong \mathbb{Z}_3$. Also $\pi_1(D) = \mathbb{F}_2 \cong \mathbb{Z} \times \mathbb{Z}$.

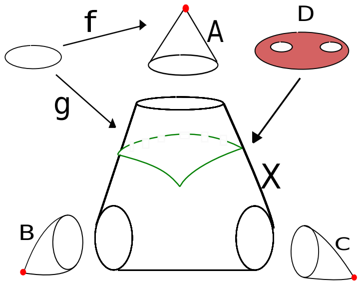


Figure 4: Pair of pants with 3 cones attached.

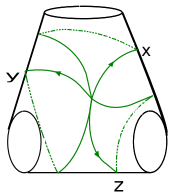


Figure 5: Pair of pants with 3 generators.

$$\pi_1 (X) \cong \langle x, y, z \mid xyz \rangle \cong \langle x, y \mid \rangle .$$

So:

$$\begin{aligned} \pi_1^{orb} \left(S^2 (3,3,3) \right) &\cong \left\langle x, y, z \mid xyz, x^3, y^3, z^3 \right\rangle \\ &\cong \left\langle x, y \mid x^3, y^3, (xy)^3 \right\rangle . \end{aligned}$$

Since $xyz = 1$ we have $z = y^{-1}x^{-1}$. So if $z^3 = 1$ then $\left(y^{-1}x^{-1}\right)^3 = 1$ so $(xy)^3 = 1$.

Exercise 15.5. Compute $\pi_1^{orb} \left(S^2 (p, q) \right)$ and $\pi_1^{orb} \left(D^2 (\overline{p}, \overline{q}) \right)$ Show that if $p \neq q$ and $\gcd (p, q) = 1$ then these groups are trivial for $\left(S^2 (p, q) \right)$ or $\mathbb{Z}_2$ for $\left(\mathbb{D}^2 (\overline{p}, \overline{q}) \right)$.

Exercise 15.6. Using van Kampen Theorem compute π^{orb} for $F = S^2 (p, q, r)$ and $D^2 (\overline{p}, \overline{q}, \overline{r})$.
Show that the covering,

$$\rho : S^2 (p, q, r) \longrightarrow \mathbb{D}^2 (\overline{p}, \overline{q}, \overline{r})$$

gives a map

$$\rho_* : \pi_1^{orb} \left(S^2 (p, q, r) \right) \longrightarrow \pi_1^{orb} \left(D^2 (\overline{p}, \overline{q}, \overline{r}) \right)$$

and the image has index two.

15.1 Seifert Fibered Spaces

There are a special class of 3-manifolds that
a) decompose as a *nice* union of disjoint circles or *equivalently*
b) are *circles bundles* over 2-orbifolds.

Exercise 15.7. Consider $S^1 \times F^2$ where F is a surface. This decomposes as a disjoint union, namely $\left\{S^1 \times \{x\} \mid x \in F\right\}$. This, then is the *trivial* S^1 bundle.

Exercise 15.8. Suppose F is a surface. Let UTF be the unit tangent bundle over$F$. This is a "twisted circle bundle"

Exercise 15.9. What 3-manifold is $UT\left(S^2\right)$?