Saul Schleimer
Italo Cipriano

18 Lecture 18

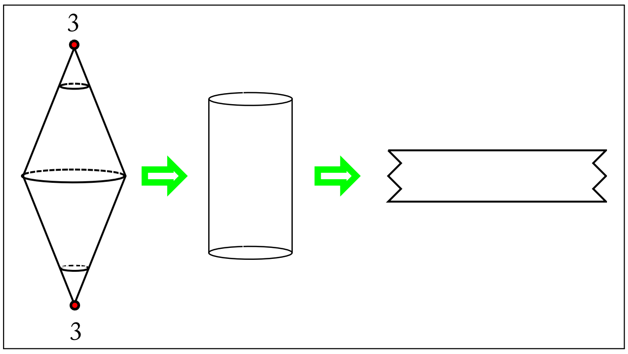
Question 18.1. Can we show that the teardrop $F = S^2(p)$ (for $p \geq 2$) is bad, by computing $\pi_1^{orb}(F)$, showing this is $\{\mathbb{1}\}$ (the trivial group) and then deducing that F has no covers other than itself?

Answer 18.2. Not really! This is because we only defined π_1^{orb} when F is good. We defined $\pi_1^{orb}(F) \doteq Deck(\rho : \tilde{F} \to F)$, where $\rho : \tilde{F} \to F$ is the universal covering (and so F is a surface). It then requires a theorem (sketched by Scott and omitted in class) that the Seifert-van Kampen procedure correctly computes a presentation for $\pi_1^{orb}(F)$.

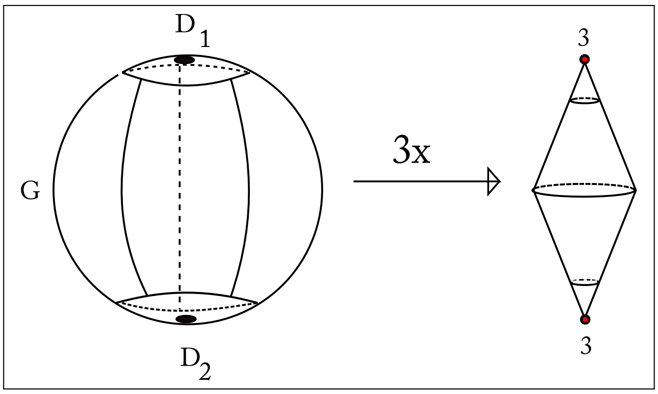
Question 18.3. Can we use Seifert-van Kampen to define $\pi_1^{orb}(F)$?

Answer 18.4. Yes, but there still work to be done!

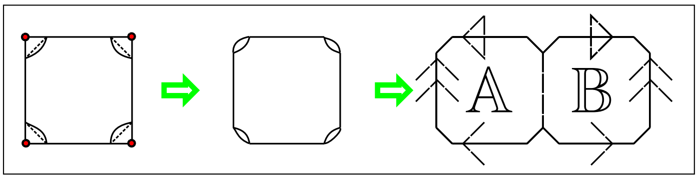
Example 18.5. Let be $F = S^2(3,3)$. Let N be a neighbourhood of all cone points. Let $F' = \overline{F - N}$, this is a surface (with boundary if $N \neq \emptyset$). Now form $\tilde{F'}$ the universal cover $\tilde{F'} = I \times \mathbb{R}$.



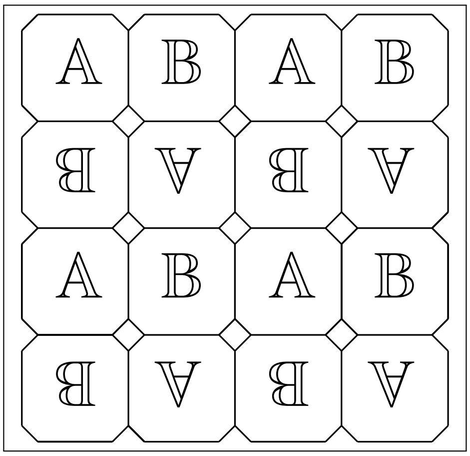
Next, find an intermediate cover $\tilde{F'} \to G \to F'$ so that we can attach a family of disks for each cone point. Last, do so an form $G \cup \{\text{Disks}\}$. In the example $G \cong I \times \mathbb{S}^1$ is an orbifold cover of F'. Each cone point gives rise to one disk, a 3-orbifold covering the neighbourhood of the point.



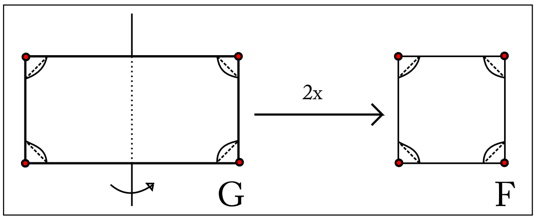
Example 18.6. Let be $F = S^2(2, 2, 2, 2)$.



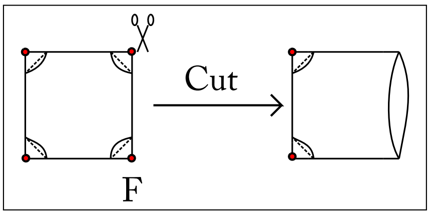
In this case the desired G is the plane minus neighbourhoods of the integer lattice points.



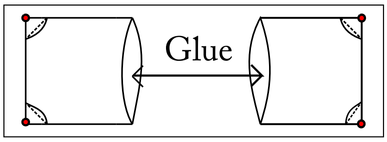
Remark 18.7. $S^2(2, 2, 2, 2)$ is double covered by a copy of itself.



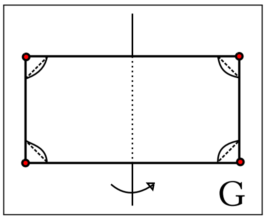
Going the other way: Cut F along its right-hand edge to obtain $D^2(2, 2)$.



Now take two copies of $D^2(2, 2)$.



Glue to get $S^2(2,2,2,2)$ the double cover.



Remark 18.8. It was promised, in the second and third week that all good orbifolds F are modelled on one of $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ as $\mathcal{X}^{\text{orb}}(F)$ is greater than, equal to, or less than zero.

Definition 18.9. Suppose M is a Seifert fibered space. Thus M is fibered by circles with the given local neighbourhoods. Define $F = M/\mathbb{S}^1$ to be the quotient $M/x \sim y$ if and only if there is a fibre containing both x and y.

Example 18.10. If $M = \mathbb{S}^1 \times F$, then $M/\mathbb{S}^1 \cong F$.

Exercise 18.11. Prove the following.

1. F has a natural orbifold structure.
2. Critical fibers with $T(p,q)$ neighbourhood are sent to cone points of order p.
3. F has no corner reflectors and no half-mirrored corners.
4. If M is closed so is F.
5. Mirrored boundary of F comes from solid Klein bottles in M.

Conclude that Seifert fibered spaces are circles bundles over orbifolds.

As an example of application of the previous exercise see problem 8.6, in which is asked to compute $F = M/\mathbb{S}^1$ when M is the three manifold given by

$$I \times \mathbb{T}^2 \Big/ (1, (p_1, p_2))) \sim (0, (1 - p_2, p_1)))\,.$$