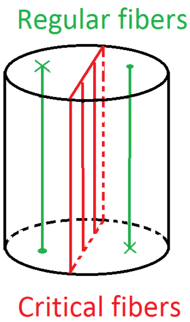
19 Lecture 19

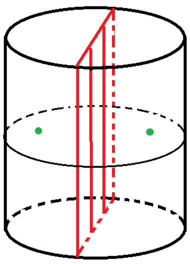
If M is equipped with a Seifert fibered structure, then $B = M/S^1$ to denote the quotient sending fibers to points of B. Call $\rho\colon M \to B$ the *Seifert bundle map*.

Example 19.1. Suppose B is a surface. For $M = S^1 \times B$ we have $\rho\colon M \to B$ defined by $(e^{i\theta}, x) = x$. This is a *product bundle*.

Example 19.2. Let K^3 be the solid Klein bottle, equipped with the usual Seifert fibered structure.

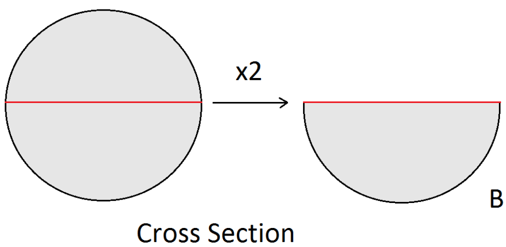


This is the solid Klein bottle $\mathbb{I} \times \mathbb{D}^2/(1,z) \sim (0,\overline{z})$ with product bundle $B = K^3/S^1$



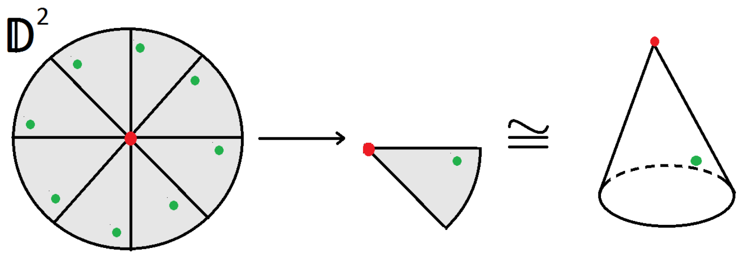
So B is the quotient of any cross section by the map $z \to \overline{z}$.

Thus B is a copy of $\mathbb{D}^2$ with two arcs on the boundary. One is a regular arc while the other is a mirrored arc, as shown below:



Example 19.3. If $M = T(p, q)$ then $B = M/S^1$ is a copy of $\mathbb{D}^2(p)$.
Recall we defined $T(p, q)$ as the quotient space $I \times \mathbb{D}^2/(1, z) \sim (0, \exp^{2\pi i \frac{q}{p}} \cdot z)$.

The cross section of a fibered solid torus, $T(p, q)$, is shown below:



With deck transformation generated by the map $z \to \exp^{2\pi i \frac{q}{p}} \cdot z$.

Exercise 19.4. Suppose $T(p, q) \cong T(p', q')$ (orientation preserving isomorphism). Show that $p = p'$ and $q = q'$.

Recall: In definition of $T(p, q)$ we required that $1 \leq q \leq p$ and $\gcd(p, q) = 1$.

Remark 19.5. Without orientation one cannot tell the difference between $\pm q($ mod $p)$. To see this, let $T = T(p, q)$ and $S = T(p, p - q)$ and define $\psi \colon T \to S$ as the map $\psi(t, z) = (t, \overline{z})$

Exercise 19.6. Check that the map ψ (defined in coordinates) gives a well-defined homeomorphism between T and S which sends fibers to fibers.

Exercise 19.7. Show that $S^1 \times \mathbb{D}^2 \cong T(1, 1)$

Exercise 19.8. Define $M(h) = I \times \mathbb{T}^2/(1, z) \sim (0, h(z))$ where $z = (x, y)$ and $h(z) = (1 - x, 1 - y)$. Show that $M(h)/S^1 \cong S^2(2, 2, 2, 2)$

Exercise 19.9. Define $r : \mathbb{T}^2 \to \mathbb{T}^2$ by $(x, y) \to (1 - y, x)$. Compute the orbifold $M(r)/S^1$ and the orbit invariants of the critical fibers of $M(r)$.

The cross section of $I \times \mathbb{T}^2$ is shown below:

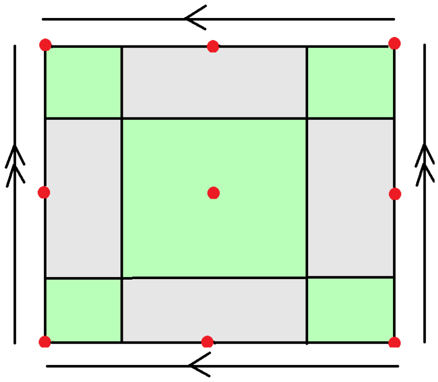
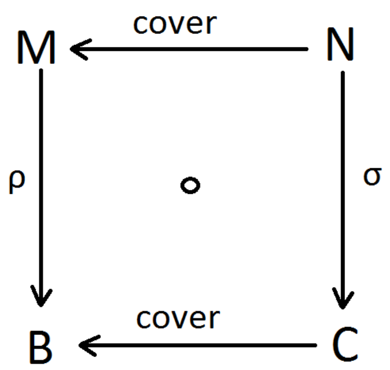


Figure 1: Cross section of $I \times \mathbb{T}^2$.

Each of the coloured squares is a copy of $T(2,1) \subseteq M(h)$.

Lemma 19.10. *Suppose $\rho\colon M \to B$ is a Seifert bundle. If N is a finite cover of M then define $C = N/S^1$. If C is a finite cover of B then there is a finite cover N of M. In either case, the following diagram commutes.*



Proof. Discussed by Scott.

Q.E.D