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2 Lecture 2

Let M^n be a triangulated n-dimensional manifold (i.e M^n is given as a union of simplices) then we define the Euler characteristic to be

$$\chi(M) = \sum_{k=0}^{\infty} (-1)^k (\# \text{ of simplices})$$

.

Example 2.1. If F is a surface with

- v-vertices
- e-edges
- f-faces

then $\chi(F) = v - e + f$.

Example 2.2. If we triangulate S^1 as the boundary of a triangle as in Figure 1 then $\chi(S^1) = 3 - 3 = 0$.

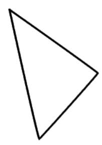


Figure 1: S^1

Example 2.3. If we triangulate S^2 as in Figure 2, then $\chi(S^2) = 4 - 6 + 4 = 2$

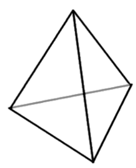


Figure 2: S^2

Exercise 2.4. Show the following:

- $\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$
- $\chi(S_g) = 2 - 2g$
- $\chi(N_c) = 2 - c$.

Theorem 2.5 (Classification of Surfaces)**.** *If F and G are compact connected 2-manifolds then F and G are homeomorphic if and only if F and G have the same*

- *orientability*
- *Euler characteristic*
- *number of boundary components.*

That is the 3 invariants; orientability, Euler characteristic and number of boundary components classify surfaces.

Exercise 2.6. Let F be the surface given as in Figure 3

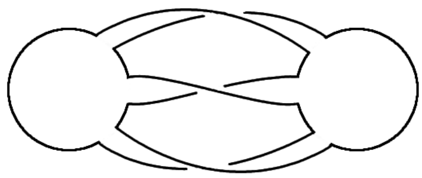


Figure 3: The surface F

Show that F is homeomorphic to a handle given by the 2-torus with an open disk removed as shown in Figure 4.

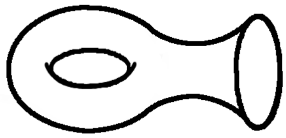


Figure 4: A handle $\mathbb{T}^2 - \mathbb{D}^\circ$

Exercise 2.7. Let G be the surface given in Figure 5, given by the quotient of an octagon.

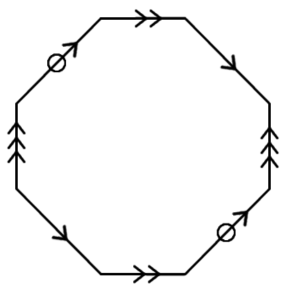


Figure 5: The surface G

Show that G is homeomorphic to S_2 as shown in Figure 6.

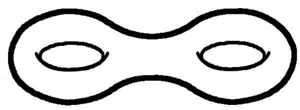


Figure 6: S_2

2.1 Orientability

Definition 2.8 (1)**.** Let M be a smooth 2-dimensional manifold. We say M is *orientable* if there exists a smooth atlas $\{(\varphi_U, U)\}$ such that if $U \cap V \neq \emptyset$ then $\varphi_V \circ \varphi_U^{-1}$ preserves the orientation of $\mathbb{R}^2$. If no such smooth atlas exists we say M is non-orientable.

Definition 2.9 (2)**.** Let F be a triangulated surface. We say that F is *orientable* if you can cyclically order the vertices of all triangles such that edge glueings reverse order.

Example 2.10. Give $\mathbb{T}^2$ the orientation as in Definition (2), as shown in Figure 7.

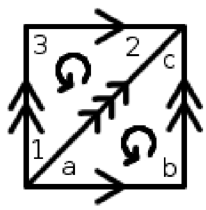
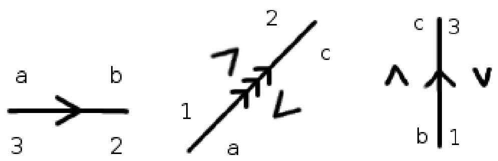


Figure 7: An orientation of $\mathbb{T}^2$: $a < b < c < a$ and $1 < 2 < 3 < 1$.

To check the definition holds we consider the edges as shown in Figure 8 and check to see they disagree.



(a) (b) (c)

Figure 8: Edges of $\mathbb{T}^2$ with orientation.

Example 2.11. Consider the Möbius band with the orientation given in Figure 9:

However we have agreement on an edge as shown in Figure 10.

Hence this triangulation does not prove the Möbius band is orientable.

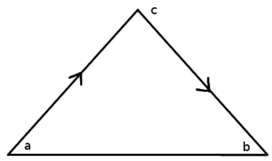


Figure 9: Möbius band with orientation: $a < b < c < a$.

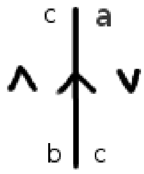


Figure 10: An edge on the M obium band with agreement.

Remark 2.12. In fact no triangulation for the Möbius band will work hence it is non-orientable, but showing this would be difficult. So to show non-orientability we make use of the following theorem.

Theorem 2.13. *A surface F is non-orientable if and only if F contains a Möbius band.*

Exercise 2.14. Show that the above definitions of orientability are equivalent. (Warning: (1) $\implies$ (2) is hard).

2.2 Double

Definition 2.15. Suppose that M is a manifold with boundary. Let $M_i = M \times \{i\}$ for $i = 0, 1$. Define the *double* of M to be

$$D(M) = M_0 \sqcup M_1/\sim \qquad (x, 0) \sim (x, 1) \Leftrightarrow x \in \partial M.$$

Example 2.16. $S^2 and \mathbb{T}^2$ can be realised as doubles as shown in Figure 11 and Figure 12 respectively.

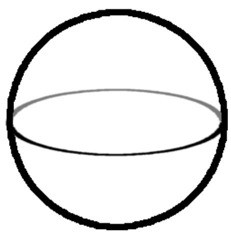


Figure 11: $S^2 = D(\mathbb{D}^2)$.

Exercise 2.17. Determine which of S_g and N_c can be realised as doubles.

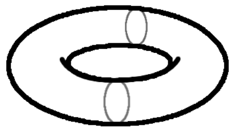
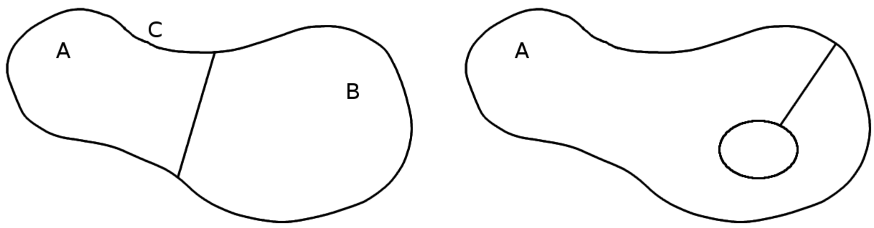


Figure 12: $\mathbb{T}^2 = D(\mathbb{A})$.

Remark 2.18. Suppose M is (a) "a nice union" or (b) "self glueing" as depicted below, then the properties given in Figure 13 hold for the Euler Characteristic.



(a) $\chi(M) = \chi(A) + \chi(B) - \chi(C)$ (b) $\chi(M) = \chi(A) - \chi(C)$

Figure 13: Manifolds with "nice union" and "self glueing".

2.3 Metric Spaces

Definition 2.19.

- A metric space (X, d_X) is *complete* if every Cauchy sequence in X converges in X.
- A function $f\colon X \to X$ is an *isometry* if for all $x, y \in X$, $d_X(x, y) = d_X(f(x), f(y))$. Let Isom(X) denote the group $\{f : X \to X \mid \text{f is an isometry}\}$.
- We say X is *homogeneous* if for all $x, y \in X$ there exists $f \in$ Isom(X) such that $f(x) = y$.
- We say X is *locally homogeneous* if for all $x, y \in X$ there exists open neighbourhoods $U, V \subseteq X$ and an isometry $f : (U, x) \to (V, y)$.

Remark 2.20. It is easy to see that if a metric space is homogeneous, then it is locally homogeneous.

Example 2.21. $\mathbb{E}^n = (\mathbb{R}^n, d_{\mathbb{E}})$ (where $d_{\mathbb{E}}$ is the standard Euclidean metric) is homogeneous.

Example 2.22. Consider the *open Möbius band* as given in Figure 14, that is

$$\mathbb{M} = \mathbb{I} \times \mathbb{R}/(1, y) \sim (0, 1 - y).$$

Note that $\mathbb{M}$ is locally homogeneous; if x, y are points of $\mathbb{M}$ then the disks of radius $1/2$ about x and about y are isometric. However, $\mathbb{M}$ is not homogeneous as no isometry can move the core curve $\mathbb{I} \times \{0\}$ off of itself.

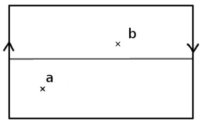


Figure 14: $\mathbb{M}$ given as a quotient space.

Exercise 2.23. Check $\text{Isom}(\mathbb{M}) \cong S^1 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Theorem 2.24 (Singer)**.** *If M is a complete, locally homogeneous metric space then it's universal cover $\tilde{M}$ is homogeneous*