# 20 Lecture 20

**Recall:** We say $M, N$ are isomorphic Seifert fibered spaces if there is a homeomorphism $f\colon M \to N$ that sends fibers to fibers.

**Thus:**$M, N$ have the same critical fibers with the same *orbit invariants* $(p, g)$ for each critical fiber $C_i \subseteq M$.

The notation $M/S^1$ is suggestive of the existence of an $S^1$-action on $M$.We then take the quotient to get the base orbifold.

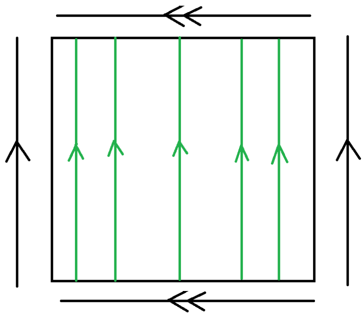
**Example 20.1.** Suppose $\mathbb{T}^3 = S^1 \times \mathbb{T}^2$. We write

$$S^1 \subset \mathbb{C}$$
$$S^1 = \left\{\exp^{i\theta} | \theta \in \mathbb{R}\right\}.$$

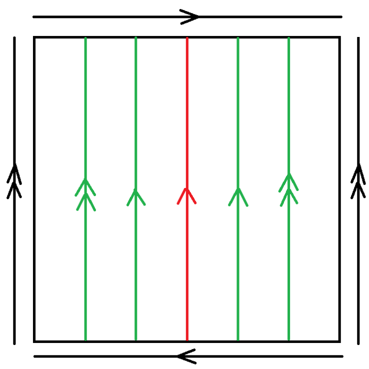
Then $S^1$ acts on $\mathbb{T}^3$ via $e^{i\theta} \cdot (z, w) = (e^{i\theta} \cdot z, w) \in S^1 \times \mathbb{T}^2$.

This suggestion is correct if the fibers of $M$ can be consistently *oriented* and incorrect otherwise.

**Example 20.2.** $\mathbb{T}^2 = S^1 \times S^1$ is a product and there is a natural action of $S^1$ on the first coordinate, as shown below:

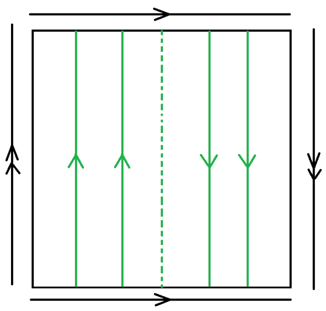


**Example 20.3.** $\mathbb{K}^2$ also has an $S^1$ action, as shown below:



Notice that rotation by $\pi$, in this example, has fixed points; namely the critical fibers.

**Example 20.4.** For the fibering of $\mathbb{K}^2$ shown below, there is no consistent orientation of the fibers. This is because $\mathbb{K}$ reverses the orientation of the central fiber.

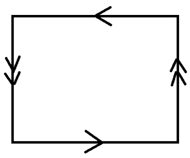


**Exercise 20.5.** If $M$ is a Seifert fibered space and there is a circle action preserving the fibers then the fibers can be consistently oriented.

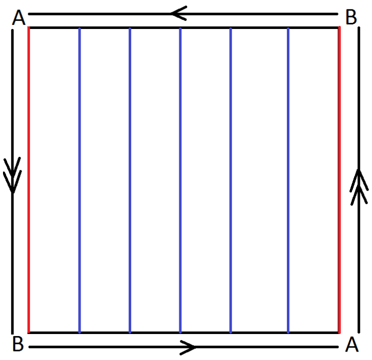
**Exercise 20.6.** Show that $F^2$, a compact surface, is fibered by circles if and only if $\chi(F) = 0$. (so $F$ is either $\mathbb{T}, \mathbb{K}, \mathbb{A}$ or $\mathbb{M}$ by the classifications)

**Question:** Can $\mathbb{P}^2$ be fibered?

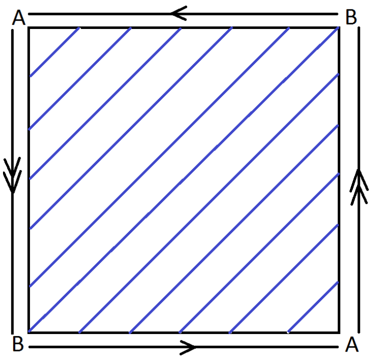
Recall that $\mathbb{P}^2$ is obtained as an identification space as shown below:



Let us try the following possible fiberings, as shown in the diagrams below:




The red line on the diagram is a line not a circle, hence this is not a fibering.




This attempt is also not a fibering because the length of the circles tends to zero at $B$.

**Example 20.7.** From example 20.2 we can see that $\mathbb{T}^2/S^1 \cong S^1$

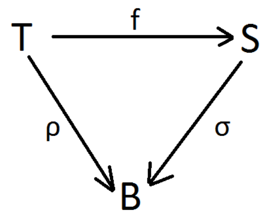
**Example 20.8.** From example 20.3 we can see that $\mathbb{K}^2/S^1 \cong I(\overline{2}, \overline{2})$

**Example 20.9.** From example 20.4 we can see that $\mathbb{K}^2/S^1 \cong S^1$

**Recall:** We say $\rho\colon T \to B$ is an *F-bundle* with the *base* $B$ and *total space* $T$ if

For every $x \in B$ there is a neighbourhood of x, $U \subseteq B$ and a *trivialization* $\phi\colon \rho^{-1}(U) \to F \times U$ such that $\rho = \text{proj}_2 \circ \emptyset$

We say two bundles $\rho\colon T \to B$ and $\sigma\colon S \to B$ are *isomorphic* if there is a homeomorphism such that the following diagram commutes:




**Example 20.10.** Up to isomorphism, $\mathbb{T}^2$ and $\mathbb{K}^2$ as in examples 20.7 and 20.9 are the only $S^1$ bundles over $S^1$.

**Fact 20.11.** If $B$ has $\pi_1(B) = \{\mathbb{1}\}$ then every $S^1$-bundle over $B$ is isomorphic to a product bundle.