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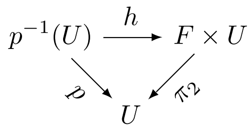
21 Lecture 21

21.1 Bundles

Suppose F, T and B are manifolds. A map $p\colon T \to B$ is an *F-bundle* if for all $x \in B$ there is a neighbourhood $U \subset B$ containing x and a map $h\colon p^{-1}(U) \to F \times U$ such that the following diagram commutes:



Here $\pi_2\colon F \times U \to U$ denotes projection to the second factor: $(f, u) \to u$.

Notation: We call F, T and B the fibre, total and base spaces respectively.

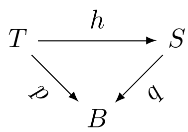
Example 21.1. Note that when $p \neq 1$, $T(p, q) \to T(p, q)/\mathbb{S}^1$ is not an $\mathbb{S}^1$-bundle even though $T(p, q)$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$. Instead, $T(p, q) \to \mathbb{D}^2$ is a *Seifert bundle*; all $\mathbb{S}^1$-bundles over surfaces are Seifert fibered spaces but the converse is not true.

Remark 21.2. If M is a Seifert fibered space and $\Sigma = \bigcup C$ is the union of all the critical fibres of M, then $M - \Sigma$ is an $\mathbb{S}^1$-bundle over $(M - \Sigma)/\mathbb{S}^1$.

Back to bundles:

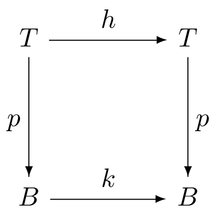
Exercise 21.3. The annulus $\mathbb{A} = I \times \mathbb{S}^1$ and Möbius band $\mathbb{M}$ are the only I-bundles over $\mathbb{S}^1$ up to isomorphism.

Definition 21.4. Two F-bundles, $p\colon T \to B$ and $q\colon S \to B$ are *isomorphic* if there is a homeomorphism $h\colon T \to S$ so that the following diagram commutes:



Exercise 21.5. The only $\mathbb{S}^1$-bundles over $\mathbb{S}^1$ up to isomorphism, are the torus $\mathbb{T}^2$ and the Klein bottle $\mathbb{K}$.

Question 21.6. Suppose $p\colon T \to B$ is an F-bundle. Given a homeomorphism $k\colon B \to B$, must there exist a homeomorphism $h\colon T \to T$ making the diagram below commute?



Answer 21.7. No. In our example we take $F = I$ and B is a pair of pants (a disk with two disks removed).

Figure 1: The space B a disk with two disks removed. Denote the central component, homeomorphic to an octagon, by O. Denote the left component by R_1 and the right component R_2.

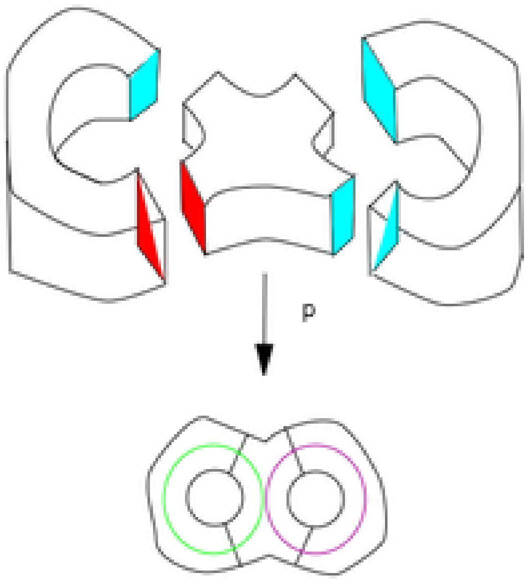


Figure 2: The space T mapped by p to the base space B with bundles $p\colon p^{-1}(O) \to O$ and $p^{-1}(R_i) \to R_i$. The green loop denotes α_1 and the purple loop denotes α_2.

Proposition 21.8. *An F-bundle over B^n is trivial.*

We defer the proof.

Remark 21.9. $\pi_1(B) = \{1\}$ does *not* suffice to imply the conclusion of Proposition 5.6.

Back to the answer: the bundles $p\colon p^{-1}(O) \to O$ and $p^{-1}(R_i) \to R_i$ are trivial.

All glueings are by translation except at the red face, where we reflect in the vertical direction.

Notice that $T|\alpha_1$ is a Möbius band and that $T|\alpha_2$ is an annulus, where α_i is the given circle in B. Thus the 180^o rotation of B about the centre of O switches α_1 and α_2. Also, there can be no $h\colon T \to T$ that lifts k.

Example 21.10. If F is a discrete space then any F-bundle is in fact a covering space.