22 Lecture 22

22.1 Example: The Hopf Fibration

Recall that $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\}$. Notice that the condition $w = 0$ determines the plane $\{(z, 0) \,|\, z \in \mathbb{C}\} \subseteq \mathbb{C}^2$.

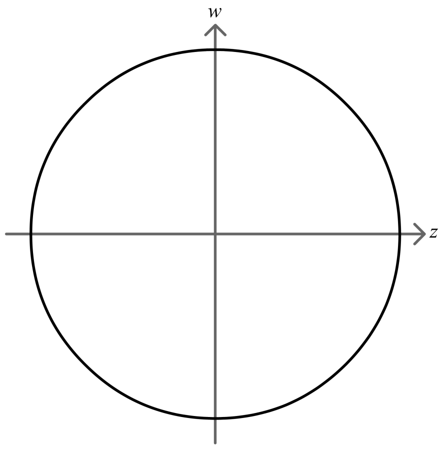

Figure 1: $\mathbb{S}^3$ drawn on complex axes z and w.

This is the same as the plane $P_{(a,b)} = \{(z, w) \,|\, az + bw = 0\}$ for $(a, b) = (0, 1)$. We claim (1) that $P_{(a,b)} \cap \mathbb{S}^3$ is a circle (i.e. an embedding of $\mathbb{S}^1$) for all $(a, b) \neq (0, 0)$. Furthermore we claim (2) that $P_{(a,b)} = P_{(c,d)}$ if and only if (a, b) is a multiple of (c, d); that is, $ad - bc = 0$.

As an example, consider $P_{(0,1)}$, the z-axis. Then we must solve the system of equations (i) $|z|^2 + |w|^2 = 1$ and (ii) $w = 0$. So $|z| = 1$ and thus

$$\mathbb{S}^3 \cap P_{(0,1)} = \left\{(e^{i\theta}, 0) \in \mathbb{C}^2 \;\middle|\; \theta \in [0, 2\pi]\right\}.$$

To see how this implies claim (1), note that for a general (a, b) there is a rotation of $\mathbb{R}^4 \cong \mathbb{C}^2$ sending $P_{(0,1)}$ to $P_{(a,b)}$ and preserving $\mathbb{S}^3$; see Figure 2.

We claim (3) that if (a, b) is not a scalar multiple of (c, d) then $P_{(a,b)} \cap P_{(c,d)} = \{0\}$; see Figure 3.

So we find that the collection

$$\{\mathbb{S}^3 \cap P_{(a,b)} \,|\, (a, b) \neq (0, 0)\}$$

is a disjoint partition of $\mathbb{S}^3$ into circles. Note that $\mathbb{S}^1$ acts on $\mathbb{S}^3$ by

$$e^{i\theta} \cdot (z, w) = (e^{i\theta} z, e^{i\theta} w)$$

and this action preserves the circles. To see this, note that the action preserves $\mathbb{S}^3$; next we check that the action preserves $P_{(a,b)}$:

$$(a, b) \cdot (e^{i\theta} z, e^{i\theta} w) = [(a, b) \cdot (z, w)]e^{i\theta} = 0.$$

Question 22.1. Can we see that $\mathbb{R}^3$ is Seifert fibred by taking the stereographic projection of the fibering of $\mathbb{S}^3$ to $\mathbb{R}^3$?

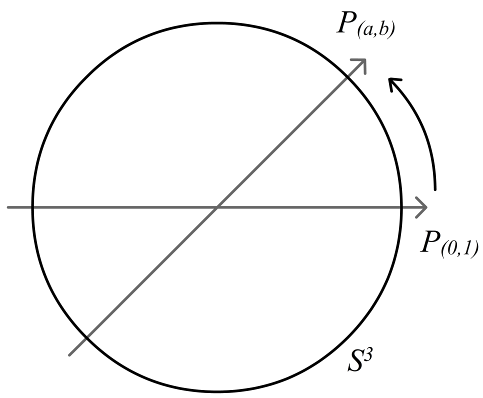

Figure 2: A rotation preserving $\mathbb{S}^3$ that sends $P_{(0,1)}$ to $P_{(a,b)}$.

Answer 22.2. No. At the point $N = (0, -1)$ stereographic projection is not defined. So the circle $\{z = 0\}$ in $\mathbb{S}^3$ is not sent to a circle.

The above partition of $\mathbb{S}^3$ is called the *Hopf fibration*. We claim (4) that the Hopf fibration is a Seifert fibering. This follows since $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{CP}^1 \cong \mathbb{S}^2$. For a more direct proof of this fact, we will consider stereographic projection.

Define the stereographic projection map $p : \mathbb{S}^3 - \{N\} \to \mathbb{R}^3$ by $x \mapsto L \cap (\mathbb{R}^3 \times \{0\})$, where L is the line through N and $x \in \mathbb{S}^3$ as in Figure 4. Figure 5 shows a picture of the image.

Notice that (c.f. claim (4)) the inside of the torus

$$T = \left\{ (z, w) \in \mathbb{C}^2 \,\middle|\, |z| = |w| = \tfrac{1}{\sqrt{2}} \right\}$$

is a copy of $T(1, 1)$. So the circle $P_{(0,1)} \cap \mathbb{S}^3$ has a fibred neighbourhood; hence by the proof of claim (1), every circle has a fibred neighbourhood. So this is indeed a Seifert fibred space. Furthermore:

(a) There are no critical fibres.

(b) The fundamental group $\pi_1(\mathbb{S}^3) = \{\mathbb{1}\}$ so also $\pi_1(\mathbb{S}^3/\mathbb{S}^1) = \{\mathbb{1}\}$ because $p_* : \pi_1(\mathbb{S}^3) \to \pi_1(\mathbb{S}^3/\mathbb{S}^1)$ is a surjection.

(c) Since $\mathbb{S}^3$ is closed, so is $\mathbb{S}^3/\mathbb{S}^1$.

We deduce from the classification of surfaces that $\mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2$.

Exercise 22.3. $\mathbb{S}^3$ has many non-isomorphic Seifert fiber structures; for example, show that $\mathbb{S}^3$ fibres over $\mathbb{S}^2(p, q)$ if $\gcd(p, q) = 1$ and $1 \leq q \leq p$.

Exercise 22.4. SO(3) is also a circle bundle over $\mathbb{S}^2$.

Remark 22.5. $\mathbb{S}^1 \times \mathbb{S}^2$, $\mathbb{S}^3$ (as above) and SO(3) are all non-isomorphic $\mathbb{S}^1$–bundles over $\mathbb{S}^2$. To see this, consider their fundamental groups.

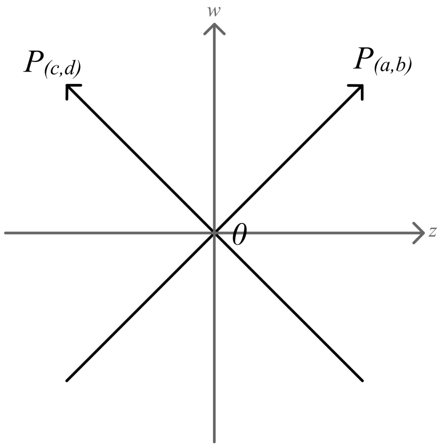

Figure 3: The intersection of $P_{(a,b)}$ and $P_{(c,d)}$ contains only the origin if $(a,b) \neq \lambda(c,d)$ for some $\lambda \in \mathbb{C} - \{0\}$.

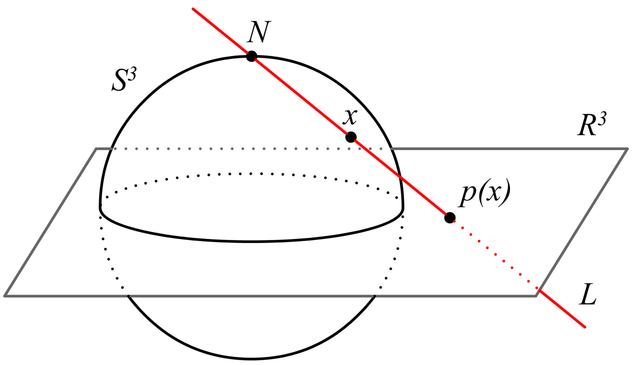

Figure 4: Stereographic projection of $\mathbb{S}^3$ to $\mathbb{R}^3$.

Definition 22.6. Let p, q be given such that $\gcd(p,q) = 1$ and $1 \leq q < p$. Define $\zeta = e^{2\pi i/p}$. Then define the *lens space*

$$L(p,q) = \mathbb{S}^3 / \sim$$

where $(z,w) \sim (z', w')$ if and only if $(z', w') = (\zeta z, \zeta^q w)$.

Exercise 22.7. $L(p,q)$ is a Seifert fibred space over $\mathbb{S}^2(p,p)$ in a natural way.

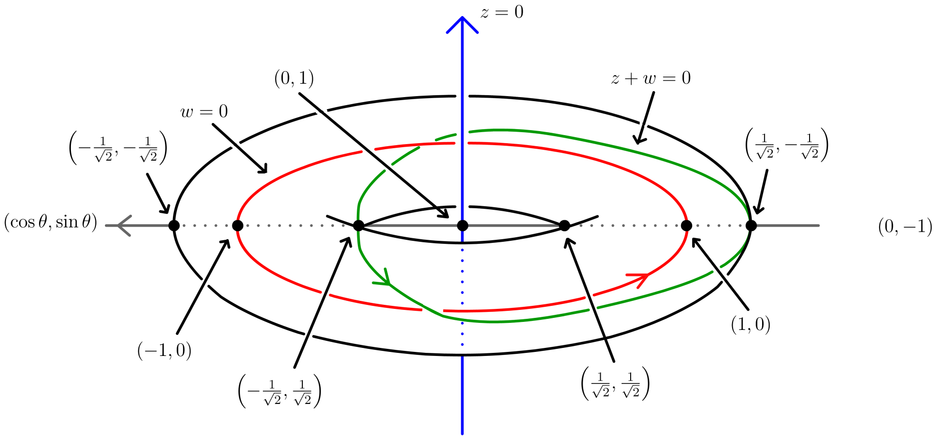

Figure 5: The image of the stereographic projection map p.