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23 Lecture 23

Recall 23.1. We were discussing the image of the stereographic projection of the Hopf fibration to $\mathbb{R}^3$.

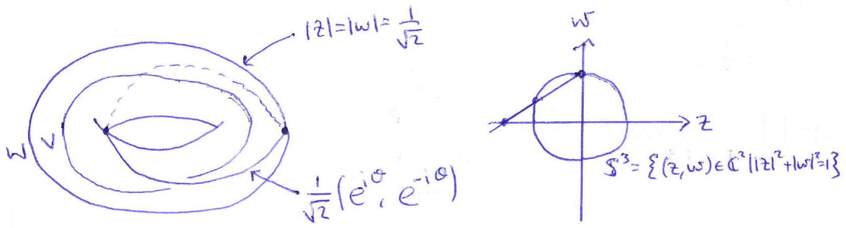


Figure 1: The torus in $\mathbb{R}^3$ is obtained as the sterogrpahic projection of the torus in $\mathbb{S}^3$

V and W are copies of $T(1,1)$.

Remark 23.2. Every solid torus $V \cong \mathbb{S}^1 \times \mathbb{D}^2$ has a *meridian* disc of the form $\{1\} \times \mathbb{D}^2$.

Picture:

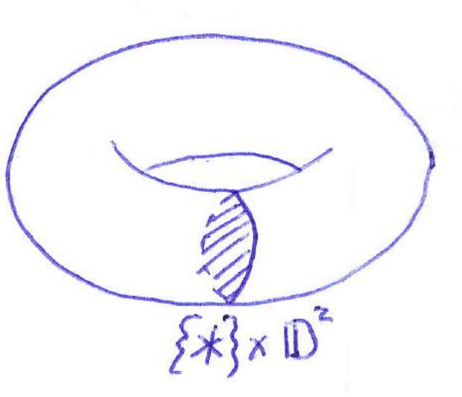


Figure 2: In the last picture we see V is fibred as $T(1,1)$

However we recall that $T(1,1)$ is isomorphic to $T(1,0)$ and in general that $T(p,q) \cong T(p,q+p)$.

Basic Topology - Embeddings

Definition 23.3. A map $f : Y \to X$ is an *embedding* if f is a homeomorphism onto its image.

Definition 23.4. An embedding $f : Y \to X$ is *proper* if X and Y are manifolds and $f^{-1}(\partial X) = \partial Y$.

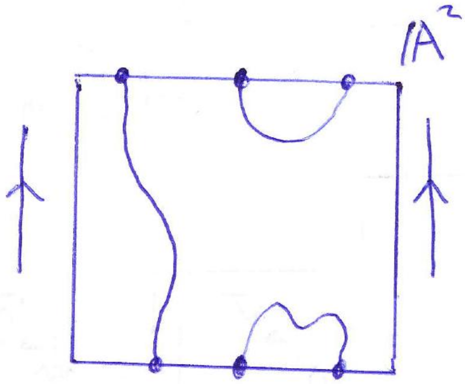


Figure 3: These are examples of proper embeddings of I into $\mathbb{A}^2$

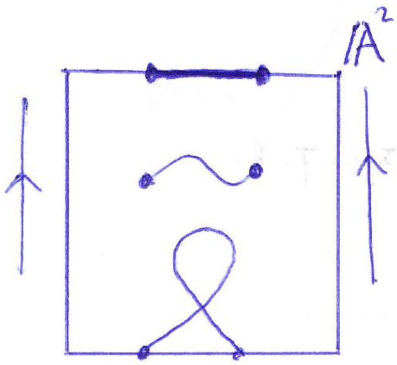
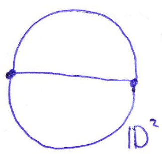


Figure 4: These are examples of embeddings of I into $\mathbb{A}^2$ which are **not** proper

Definition 23.5. Suppose $F : Y \times I \to X$ and define $f_t(y) = F(y, t)$. Then F is a *proper isotopy* if f_t is an embedding for all $t \in I$.

We think of F as a 1-parameter family of embeddings or in other words a continuous motion of Y in X.

Exercise 23.6. Up to proper isotopy there is only **one** properly embedded arc in $\mathbb{D}^2$.



Hint: Look up Alexander's trick.

Exercise 23.7. There are three properly embedded arcs in $\mathbb{A}^2$ (and two in $\mathbb{M}^2$) up to proper isotopy.

Hint: Don't waste pages thinking about transversality.

Exercise 23.8. Suppose M is a Seifert-fibred space in which all the fibres can be consistently oriented. Show all regular fibres are isotopic. (NB: not properly)

Remark 23.9. If fibres can't be consistently oriented then regular fibres are isotopic as sets but not as parameterised circles.

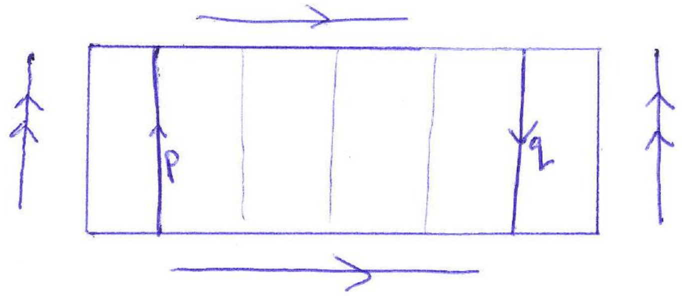


Figure 5: Fibres which are isomorphic as sets but not as functions

As sets the two fibres are the same and they are given by

$$p : \mathbb{S}^1 \to \mathbb{T}^2 \quad p(e^{i\theta}) = (e^{i\theta}, x)$$
$$q : \mathbb{S}^1 \to \mathbb{T}^2 \quad q(e^{i\theta}) = (e^{-i\theta}, y)$$

As functions they are different.

Remark 23.10. If $\mathbb{M}$ contains a vertical $\mathbb{K}^2$ (consisting of a union of fibres) then all regular fibres with any orientation are isotopic.

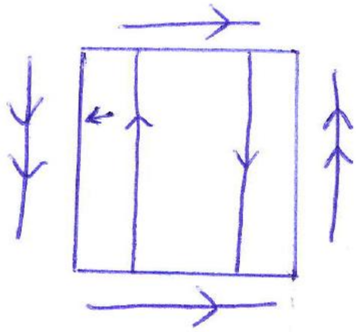
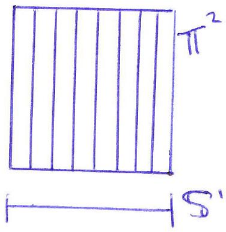


Figure 6: A vertical Klein bottle

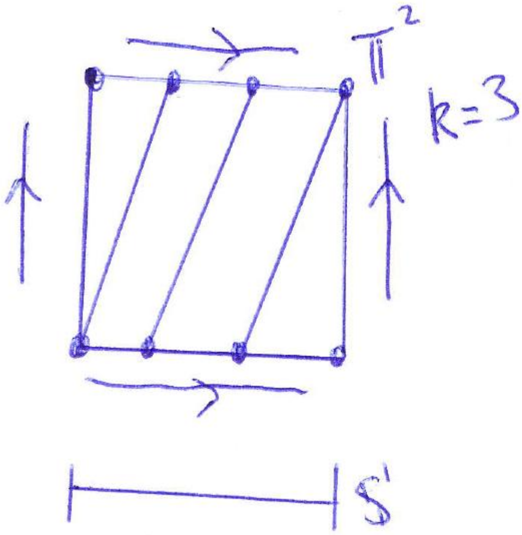
Sections

Definition 23.11. Suppose $F \to T \xrightarrow{p} B$ is an F-bundle over B. A map $\sigma : B \to T$ is called a *section* of p if $p \circ \sigma = id_B$.

Example 23.12. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.



Then $\sigma_k : \mathbb{S}^1 \to \mathbb{T}^2$ where $e^{i\theta} \mapsto (e^{ki\theta}, e^{i\theta})$ is a section.



Exercise 23.13. All sections of p are isotopic to σ_k for some (unique) k.

Exercise 23.14. For every section σ of $p : \mathbb{T}^2 \to \mathbb{S}^1$ there is an automorphism $g \in Aut(\mathbb{T}^2 \xrightarrow{p} \mathbb{S}^1)$ such that $g \circ \sigma = \sigma_0$. [Important: Give g explicitly for all of the σ_k]

Exercise 23.15. Do the above exercise for $p : \mathbb{K}^2 \to \mathbb{S}^1$.

There are two sections up to isotopy but only one up to isomorphism.

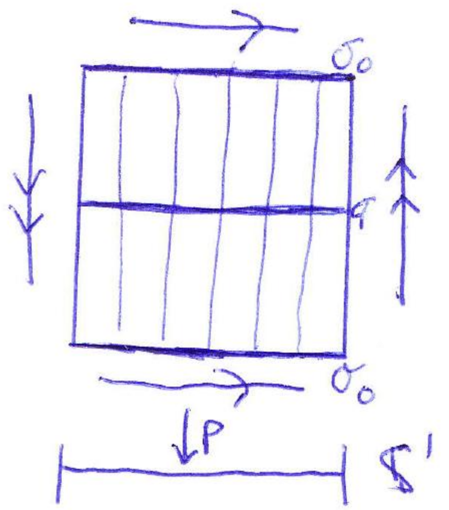


Figure 7: The two sections (up to isotopy) of $p : \mathbb{K}^2 \to \mathbb{S}^1$