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MA4J2
6th March 2012

24 Lecture 24

Question 24.1. How do you classify the surfacs admitting a fibration by circles?

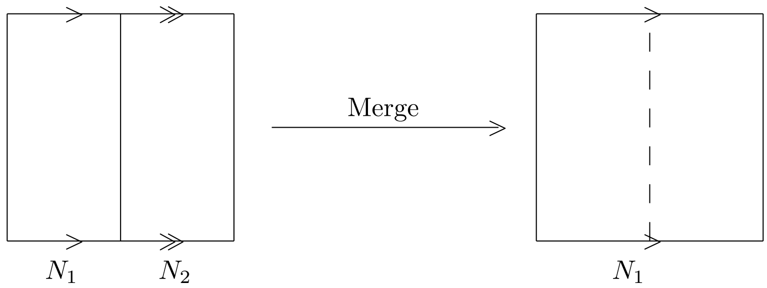
Answer 24.2. Suppose F^2 is fibred by circles, so every $C \subset F$ has a fibred neighborhood $N(C) \cong \mathbb{S}^1 \times I$ or $N(C) \cong \mathbb{M}$, the Möbius band.

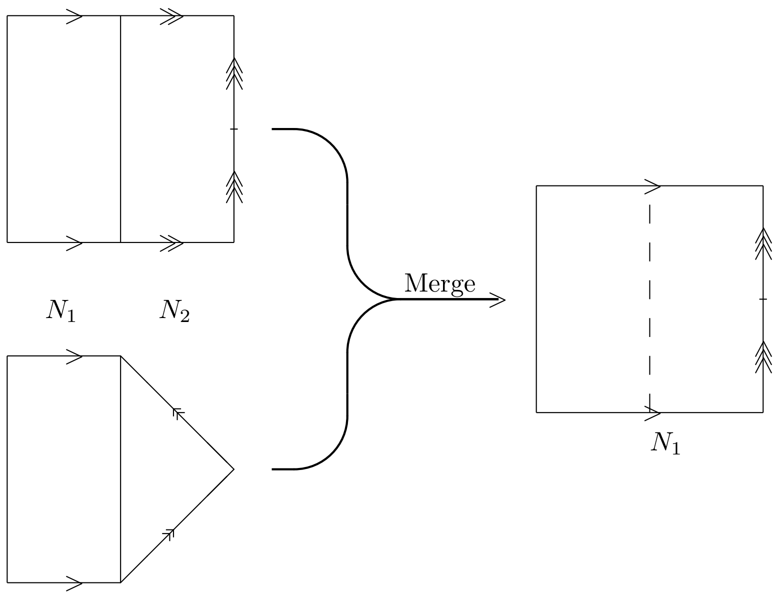
Lets also suppose F is compact, so then there is a finite collection of fibres C_i such that $N_i = N(C_i)$ cover F.

We may suppose that each N_i and N_j are disjoint or intersect only in at most two fibres (ie on their boundary).

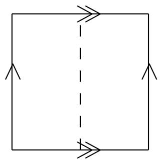
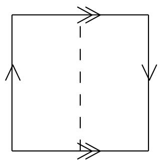
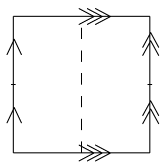
Also, if $N_i \cap N_j$ is a single fibre and N_i is an annulus then $N_i \cup N_j \cong N_j$ so we can merge them.

Finally F is a union of at most two N_is, since if there were 3 or more then some N_i has two boundary components so is an annulus and may be merged as above.





Finally list cases

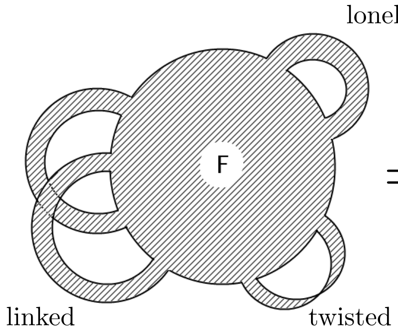
1 annulus	$\mathbb{A}$
1 Möbius band	$\mathbb{M}$
~~1 annulus and 1 Möbius band~~	This merges to the second case
2 annuli ($\mathbb{S}^1$ bundles over $\mathbb{S}^1$)	 $\mathbb{T}^2$  $\mathbb{K}^2$
2 Möbius bands (Orbifold $\mathbb{S}^1$ bundles over $I(\overline{2},\overline{2})$)	

$\mathbb{S}^1$-bundles over surfaces

Disks with handles

Suppose F is compact, connected and $\partial F \neq \emptyset$.

Exercise 24.3. F is homeomorphic to a disk with bands/handles attached.

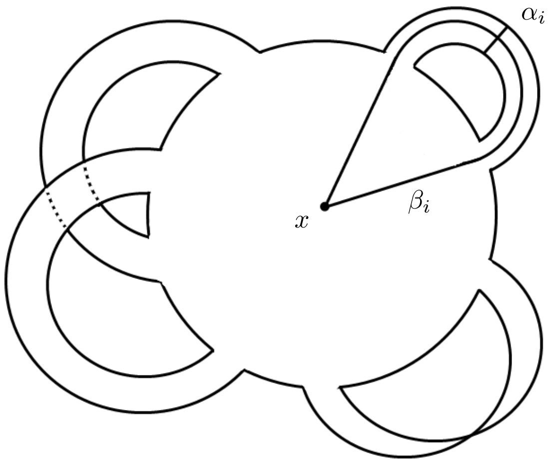


$$= \frac{\mathbb{D}\sqcup\{I^2_\alpha|\alpha\in A\}}{\text{glue ends to } \partial\mathbb{D}}$$

Proof. Use classification theorem for surfaces. □

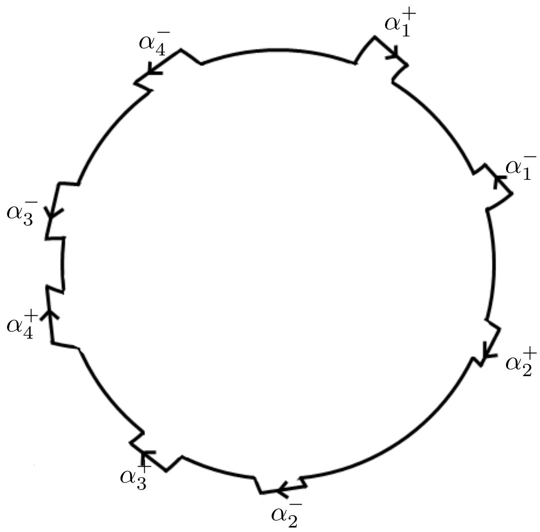
Let $F = \mathbb{D}^2 \cup \{\text{handles}\}$. Let $\alpha = \{\alpha_i\}$ be a collection of arcs, one per handle, cutting the handles open so that $F \setminus n(\alpha) \cong \mathbb{D}^2$, where $n(\alpha)$ is an open regular neighborhood of α.

Fix $x \in \mathbb{D}^2$ a basepoint. For each i, pick a loop β_i such that $\alpha_i \cap \beta_i$ is a single point and $\alpha_i \cap \beta_j = \emptyset$ if $i \neq j$.



Exercise 24.4. $\{\beta_i\}$ generates $\pi_1(F, x)$.

The disc $D = F \setminus n(\alpha)$ has arcs on its boundary $\{\alpha_i^\pm\}$.



$\mathbb{S}^1$-bundles over F

Suppose $p : T \to F$ is an $\mathbb{S}^1$ bundle. Let $T|_\Omega$ be the restriction of T to $\Omega \subset F$, namely the subbundle of T lying above Ω.

Notice that for all i, $T|_{\alpha_i}$ is an annulus (the only $\mathbb{S}^1$-bundle over I). Also, $T|_{\beta_i}$ is either a torus or a Klein bottle.

Thus there is a function

$$\tau : \pi_1(F, x) \to \mathbb{Z}_2 = \{\pm 1\}$$

where

$$\tau(\gamma) = \begin{cases} +1 & \text{if } T|_\gamma \text{ is a torus} \\ -1 & \text{if } T|_\gamma \text{ is a Klein bottle} \end{cases}$$

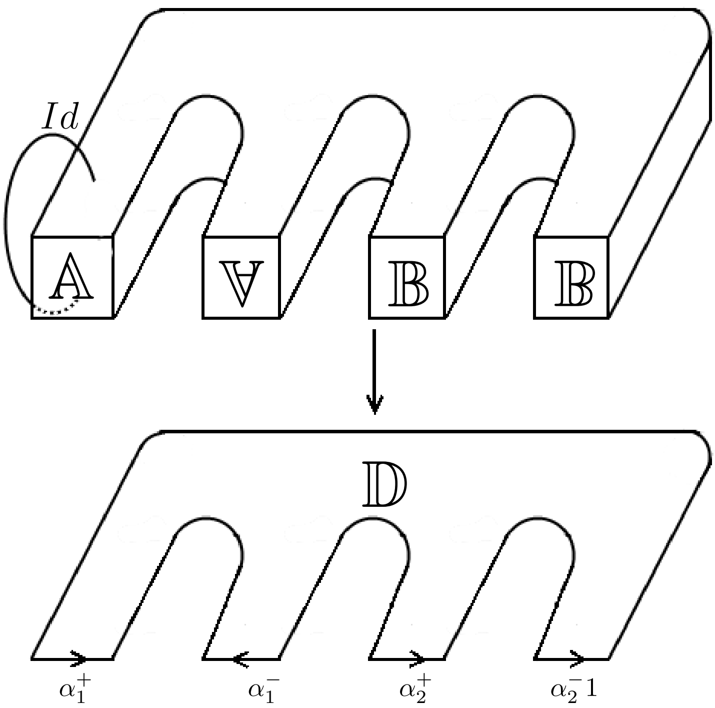
Remark 24.5. We a really using $T|_\gamma$ to represend the pull-back bundle.

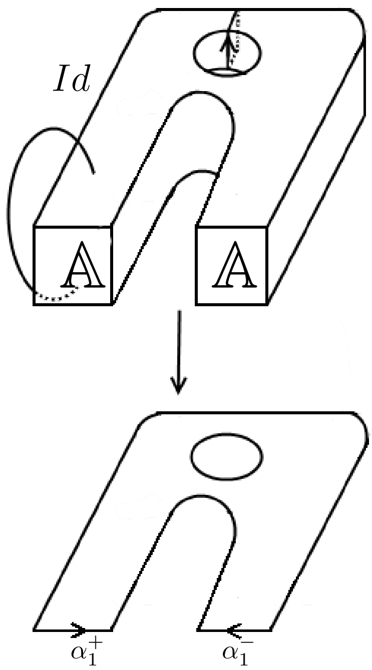
So [check] this $\tau = \tau_p$ is a homomorphism of groups.

Exercise 24.6. $p : T \to F$ determines and is determined by $\tau_p : \pi_1(F, x) \to \mathbb{Z}_2$, up to isomorphism.

[We are always assuming that $\partial F \neq \emptyset$.]

Pictures of T





$$\cong \mathbb{S}^1 \times (\text{pants})$$