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25 Lecture 25

Question 25.1. What do you get if you cut a Mobius band along the central curve? What if you cut along a curve one third away from the boundary?

Question 25.2. Suppose F is a compact connected surface with boundary given as a disk with handles. Compute $\pi_1(F, x)$.

Question 25.3. What is cutting?

Answer 25.4. Suppose $Y \subset X$. We say that $X - n(Y)$ is the result of cutting X along Y.

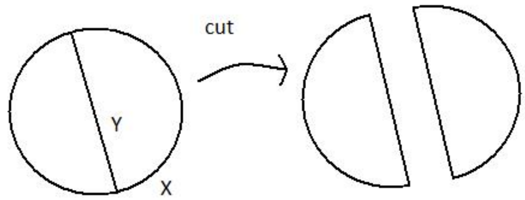

Figure 1: Cutting X along Y.

Plan for the rest of the course:
We will define the *Euler number* of an S^1-bundle over a surface I and more generally of a Seifert fibered space $M \to M/S^1$.
We will use this to give presentations of $\pi_1(M, x)$ for M a Seifert fibered space.

From last time:
F is a disk with handles, $\{\alpha_i\}$ is a collection of arcs cutting open the handles, $\{\beta_j\}_{j=1}^k$ is a collection of loops based at $x \in F$ such that:

$$\beta_i \cap \beta_j = x \text{ if } i \neq j$$

$$|\alpha_i \cap \beta_j| = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

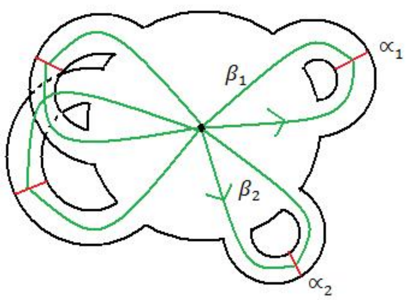

Figure 2: An example of F with corresponding $\{\alpha_i\}$ and $\{\beta_j\}$.

Suppose $p : T \to F$ is an S^1-bundle.

Definition 25.5. Define $\tau : \pi_1(F, x) \to \{\pm 1\}$ by:

$$\tau(\beta_i) = \begin{cases} +1 & \text{if } \tau \mid \beta_i \cong \mathbb{T}^2 \\ -1 & \text{if } \tau \mid \beta_i \cong \mathbb{K}^2 \end{cases}$$

Example 25.6.

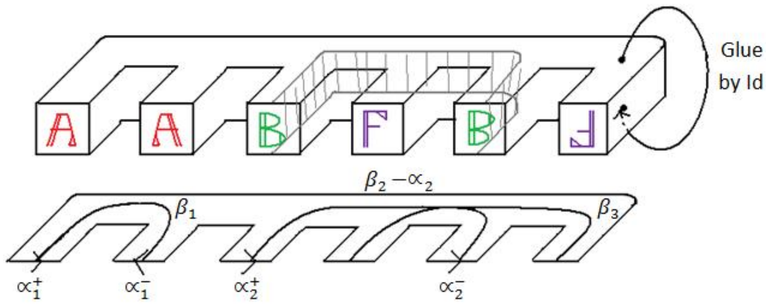

Figure 3: Here $\tau(\beta_1) = 1, \tau(\beta_2) = -1, \tau(\beta_3) = -1$.

We extend τ to the rest of $\pi_1(F, x)$ multiplicatively, i.e.

$$\tau(\gamma \cdot \delta) = \tau(\gamma) \cdot \tau(\delta)$$

and use the fact that $\{\beta_j\}$ is a free generating set for the free group $\pi_1(F, x) \cong \mathbb{F}_k$.

For proof that $\pi_1(F, x)$ is free, see figure 4.

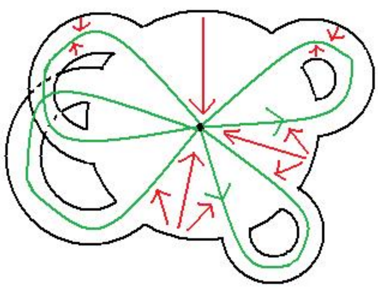

Figure 4: F deformation retracts to $\cup_{j=1}^{k} \beta_j$.

Exercise 25.7. Given $p : T \to F$ and τ_p as above, τ_p determines T up to isomorphism (of S^1-bundles over F).

Definition 25.8. Suppose $\rho : \pi_1(F, x) \to \mathbb{Z}^2$ is given by:

$$\rho(\beta_j) = \begin{cases} +1 & \text{if the } j\text{'th handle is not twisted} \\ -1 & \text{if the } j\text{'th handle is twisted} \end{cases}$$

There is (thus) a unique $T \to^p F$ such that $\tau_p = \rho$. This T is called the *orientation* S^1-bundle over F. It is the only S^1-bundle over$F$ that is orientable as a three manifold.

Exercise 25.9. Verify this.

Claim 25.10. The bundle $T \to^p F$ has a *section* $\sigma : F \to T$.

To prove: As in figure 3 (because $\partial F \neq \phi$), use the central cross section.

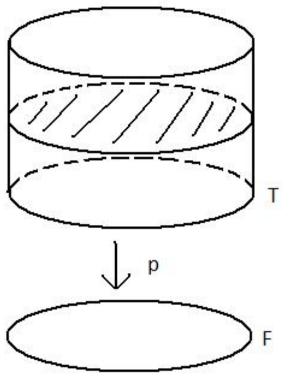

Figure 5